An Exact Price Index for the Almost Ideal Demand System

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Abstract — We show that an exact price index for the Almost Ideal Demand System can be evaluated using data on expenditure shares and prices at two comparison points, and at their geometric mean.

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1. Introduction

Deaton and Muellbauer (1980) introduced the Almost Ideal Demand System (AIDS), which satisfies a number of desirable theoretical properties, and is very convenient to estimate. For welfare comparisons, it would be useful to know the corresponding exact price index. We derive an exact index by making use of the associated Divisia index, defined as the expenditure-weighted integral of the change in prices along a path between two points. We first show that the Divisia index can be measured using data on just the prices and expenditure shares at the two endpoints, and the midpoint, of this path. We then show that the Divisia index exactly measures the change in expenditure needed to obtain a constant level of utility at the two price points. Thus, a convenient and exact measure of the price index for the AIDS is obtained.

2. Almost Ideal Demand System

Denoting the vector of prices by $\mathbf{p}=(p_1,\ldots,p_n)$, the expenditure needed to obtain a utility level of u under the AIDS is given by:

$$\log e(\mathbf{p},u)=a_0+\sum_{i=1}^{n}\alpha_i \log p_i+\tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}\log p_i \log p_j+u\beta_0\prod_{k=1}^{n}p_k^{\beta_k}\ , \qquad (1)$$

where without loss of generality we set $\gamma_{ij}=\gamma_{ji}$. The restrictions $\sum_{i=1}^{n}\alpha_i=1$ and $\sum_{j=1}^{n}\gamma_{ij}=\sum_{k=1}^{n}\beta_k=0$ ensure that $e(\mathbf{p},u)$ is homogeneous of degree one in $\mathbf{p}$. Deaton and Muellbauer (p. 313) show that the expenditure shares are given by:

$$w_j=\alpha_j+\sum_{i=1}^{n}\gamma_{ij}\log p_i+\beta_j\log(Y/P)\ , \quad j=1,\ldots,\text{n}, \qquad (2)$$

where Y denotes total expenditure, and P is an "aggregate" of the prices defined by:

$$\log P=a_0+\sum_{i=1}^{n}\alpha_i\log p_i+\tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}\log p_i\log p_j\ . \qquad (3)$$

Notice that good j is a luxury (necessity) as β_j >(<) 0, with an expenditure share rising (falling) in income, so the underlying utility function is non-homothetic.

2. Divisia Price Index

The Divisia price index is defined as an expenditure-weighted integral of price changes, between an initial and final point. Let $\mathbf{p}_t$: $[0,1] \rightarrow \Re^n_{++}$ be a function of t that describes a piecewise smooth path for prices from $\mathbf{p}_0$ to $\mathbf{p}_1$. Let Y_t define a path for income, and let $\mathbf{w}_t$ equal $w(\mathbf{p}_t, Y_t)$, where $w(\mathbf{p},Y)$ is a function giving the vector of expenditure shares that occurs at prices $\mathbf{p}$ and expenditure level Y. Letting $\dot{\mathbf{p}}_t$ denote the vector of log price changes $(\partial p_{it}/\partial t)\ /p_{it}$, the Divisia index is then expressed as:

$$\log P_D \;=\; \int_0^1 \mathbf{w}_t \cdot \dot{\mathbf{p}}_t \; dt\,. \tag{4}$$

As shown by Samuelson and Swamy (1974), the Divisia index in (4) is dependent on the path chosen for prices whenever the underlying utility function is non-homothetic. We shall choose a particular "log-linear" path given by: $\log \mathbf{p}_t = \log \mathbf{p}_0 + t(\log \mathbf{p}_1 - \log \mathbf{p}_0)$, and $\log Y_t = \log Y_0 + t(\log Y_1 - \log Y_0)$. Substituting these into (2) and (3), it is immediate that the expenditure shares w_j are *quadratic* functions of t. Simpson's rule therefore implies that the integral for P_D can be evaluated by putting a two-thirds weight on the expenditure shares implied by the *geometric mean* of the initial and final values for prices and income. In particular,

$$\log P_D = [\tfrac{1}{6}\, w(\mathbf{p}_0, Y_0) + \tfrac{2}{3}\, w(\mathbf{p}_{0.5}, Y_{0.5}) + \tfrac{1}{6}\, w(\mathbf{p}_1, Y_1)] \cdot \log(\mathbf{p}_1/\mathbf{p}_0), \tag{5}$$

where $\log(\mathbf{p}_1/\mathbf{p}_0)$, which equals $\dot{\mathbf{p}}_t$, denotes a vector with elements of the form $\log(p_{i1}/p_{i0})$. This expression also gives a good approximation for $\log P_D$ for almost any other demand model when the path is straight line in log price and log income space.

3. Equality to an Exact Price Index

Exact (or "economic") price indexes reflect changes in the expenditure function $e(\mathbf{p}, u_r)$ between an initial price vector $\mathbf{p}_0$ and final vector $\mathbf{p}_1$, at constant "reference" utility u_r:

$$P_E(\mathbf{p}_0, \mathbf{p}_1, u_r) \;=\; \frac{e(\mathbf{p}_1, u_r)}{e(\mathbf{p}_0, u_r)}\ . \tag{6}$$

Corresponding to the price and income paths are the actual utility levels u_t, satisfying $Y_t = e(\mathbf{p}_t, u_t)$ for $t \in [0,1]$.

We need to establish how the Divisia price index, which is conveniently measured as in (5), is related to the underlying exact price index. Reinsdorf (1998) provides general results on the relation between the Divisia and exact price indexes in the non-homothetic case. For the special AIDS case, we can proceed by setting $Y_t = e(\mathbf{p}_t, u_t)$, and, using (1)-(3), rewrite the shares as,

$$w_{jt} = \alpha_j + \sum_{i=1}^{n} \gamma_{ij} \log p_{it} + u_t \beta_0 \beta_j \prod_{k=1}^{n} p_{kt}^{\beta_k}\ . \tag{3'}$$

Substituting this into the Divisia index (4), we obtain:

$$\log P_D \;=\; \int_0^1 \left[\boldsymbol{\alpha} + \sum_{i=1}^{n} \gamma_{ij} \log \mathbf{p}_t + u_t \beta_0 \prod_{k=1}^{n} p_{kt}^{\beta_k} \boldsymbol{\beta} \right] \cdot \dot{\mathbf{p}}_t \; dt\ , \tag{7}$$

where $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_n)$ and $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_n)$. Now define a reference level of utility as:

$$u_r \;\equiv\; \int_0^1 \left[u_t \beta_0 \prod_{k=1}^{n} p_{kt}^{\beta_k} \boldsymbol{\beta} \right] \cdot \dot{\mathbf{p}}_t \; dt \Bigg/ \int_0^1 \left[\beta_0 \prod_{k=1}^{n} p_{kt}^{\beta_k} \boldsymbol{\beta} \right] \cdot \dot{\mathbf{p}}_t \; dt\ , \tag{8}$$

where the time-dependent utility level u_t is absent from the denominator, which we assume is non-zero. Then it is immediate that:

$$\log P_D \;=\; \int_0^1 \left[\boldsymbol{\alpha} + \sum_{i=1}^{n} \gamma_{ij} \log \mathbf{p}_t + u_r \beta_0 \prod_{k=1}^{n} p_{kt}^{\beta_k} \boldsymbol{\beta} \right] \cdot \dot{\mathbf{p}}_t \; dt\ . \tag{9}$$

Comparing the integrand in (9) with (3'), it is evident that the term in brackets equals the expenditure shares obtained with price $\mathbf{p}_t$ and *constant* utility u_r. These are equal to the derivatives of log $e(\mathbf{p}_t, u_r)$ wrt. log $\mathbf{p}_t$. Then by direct integration of (9), it follows that:

$$\log P_D = \log e(\mathbf{p}_1, u_r) - \log e(\mathbf{p}_0, u_r) = \log P_E(\mathbf{p}_0, \mathbf{p}_1, u_r)\ . \qquad (10)$$

Thus, we have shown that the Divisia index equals the exact price index, evaluated at the reference utility level given by (8). We can interpret this reference utility by making use of the fact that log-linear paths $\log \mathbf{p}_t = \log \mathbf{p}_0 + t(\log \mathbf{p}_1 - \log \mathbf{p}_0)$ imply that $\dot{\mathbf{p}}_t$ is a vector of constants. Hence $\boldsymbol{\beta}\cdot\dot{\mathbf{p}}_t$ is constant, and canceling from the numerator and denominator of (8) gives, $u_r = \int_0^1 u_t \delta_t \, dt \Big/ \int_0^1 \delta_t \, dt$, where $\delta_t \equiv \prod_{k=1}^{n} p_{kt}^{\beta_k} > 0$. The reference utility is therefore a *weighted average* of the utilities obtained along the log-linear price and income path.

4. Conclusions

The AIDS provides a flexible description of non-homothetic consumer tastes. The definition of an exact price index requires a reference utility level, while the Divisia price index requires a path to be chosen for prices and income. We have shown that these twin requirements are intimately linked via (8): given any price and income path (and implied path for utility), the reference utility level defined there allows the Divisia index to equal the exact price index, as in (10). For the particular case where the path is a straight line in log price and log income space, the Divisia index is conveniently measured by (5), and the reference utility is a weighted average of the utilities along the path. It is hoped that formula (5) will prove useful in empirical work.

References

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