Should Exact Index Numbers have Standard Errors?

Theory and Application to Asian Growth

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APPENDIX

Proof of Proposition 1

(*i*) First, we will derive the conventional Sato-Vartia price index. Rewrite the unit-cost function for constant values of $b_{it} = b_i$ as,

$$c(\mathbf{p}_t, \mathbf{b}) = \left(\sum_{i=1}^{N} b_i \, p_{it}^{\gamma} \right)^{1/\gamma} . \tag{A1}$$

where $\gamma = 1 - \eta$. The cost shares are,

$$s_{i\tau} = \partial \ln c(\mathbf{p}_\tau, \mathbf{b}) / \partial \ln p_{i\tau} = c(\mathbf{p}_\tau, \mathbf{b})^{-\gamma} b_i \, p_{i\tau}^{\gamma}, \tag{A2}$$

for τ = t-1,t. Rewriting these, we obtain,

$$\frac{c(\mathbf{p}_t, \mathbf{b})}{c(\mathbf{p}_{t-1}, \mathbf{b})} = \frac{p_{it} s_{it-1}^{1/\gamma}}{p_{it-1} s_{it}^{1/\gamma}}, \qquad \text{for } i = 1,\ldots,N. \tag{A3}$$

Take a geometric mean of (A3) using the weights w_i from (8) to obtain:

$$\frac{c(\mathbf{p}_t, \mathbf{b})}{c(\mathbf{p}_{t-1}, \mathbf{b})} = \prod_{i=1}^{N} \left(\frac{p_{it}}{p_{it-1}} \right)^{w_i} \prod_{i=1}^{N} \left(\frac{s_{it-1}}{s_{it}} \right)^{w_i / \gamma} = \prod_{i=1}^{N} \left(\frac{p_{it}}{p_{it-1}} \right)^{w_i} , \tag{A4}$$

which shows that the Sato-Vartia index on the right of (A4) equals the ratio of unit-costs on the left. To show that the product of share terms in the center of (A4) equals unity, take its logarithm to obtain:

$$\sum_{i=1}^{N} w_{it}(-\Delta \ln s_{it})/\gamma = \frac{-(1/\gamma)\sum_{i=1}^{N}\Delta s_{it}}{\sum_{i=1}^{N}(\Delta s_{it}/\Delta \ln s_{it})} = 0,$$

where the first equality follows from the definition of w_i in (8), and the second equality follows from the fact that the cost shares $s_{i\tau}$ sum to unity over i =1,…,N, for $\tau = t–1$, t.

(ii) Next, we show that we can choose the $\tilde{b}_i$ such that:

$$\frac{c(\mathbf{p}_t, \tilde{\mathbf{b}})}{c(\mathbf{p}_{t-1}, \tilde{\mathbf{b}})} = \prod_{i=1}^{N}\left(\frac{p_{it}}{p_{it-1}}\right)^{w_i}, \qquad \text{(A5)}$$

where the weights w_i are evaluated as in (8) using the cost shares $s_{i\tau} = \partial \ln c(\mathbf{p}_\tau, \mathbf{b}_\tau)/\partial \ln p_{i\tau}$ for $\tau =$ t–1, t. From (A4), the ratio of unit-costs on the left of (A5) equals:

$$\frac{c(\mathbf{p}_t, \tilde{\mathbf{b}})}{c(\mathbf{p}_{t-1}, \tilde{\mathbf{b}})} = \prod_{i=1}^{N}\left(\frac{p_{it}}{p_{it-1}}\right)^{\tilde{w}_i}, \qquad \text{(A6)}$$

where the $\tilde{w}_i$ are calculated as in (8) but using the cost shares $\tilde{s}_{i\tau} = \partial \ln c(\mathbf{p}_\tau, \tilde{\mathbf{b}})/\partial \ln p_{i\tau}$, $\tau = t–1$, t.

Thus, a sufficient condition for (A5) to hold is that there exist $\tilde{b}_i$ such that:

$$w_i = \tilde{w}_i \quad i = 1, ..., N. \qquad \text{(A7)}$$

From the definition of the weights in (8), condition (A7) will hold iff there exists $k_1 > 0$ such that,

$$\frac{\Delta s_{it}}{\Delta \ln s_{it}} = k_1\left(\frac{\Delta \tilde{s}_{it}}{\Delta \ln \tilde{s}_{it}}\right), \; i = 1, ..., N. \qquad \text{(A8)}$$

Define $\pi_t \equiv c(\mathbf{p}_t, \tilde{\mathbf{b}}) / c(\mathbf{p}_{t-1}, \tilde{\mathbf{b}})$. Then, from (A2), the denominator on the right of (A8) equals $\gamma(\Delta\ln p_{it} - \ln \pi_t)$. (If this is zero then we can replace the bracketed term on the right side of (A8) by its limiting value of $\tilde{s}_{it-1} = \tilde{s}_{it}$ and adapt what follows to solve for $\tilde{b}_i$. So without loss of generality, suppose $\gamma(\Delta\ln p_{it} - \ln \pi_t) \neq 0$.) Also using (A1) and (A2) to substitute for the numerator on right side of (A8), we have,

$$\left(\frac{\Delta s_{it}}{\Delta \ln s_{it}}\right)(\Delta \ln p_{it} - \ln \pi_t) = \frac{k_1}{\gamma}\left(\frac{\tilde{b}_i p_{it}^{\gamma}}{\sum_{j=1}^{N} \tilde{b}_j p_{jt}^{\gamma}} - \frac{\tilde{b}_i p_{it-1}^{\gamma}}{\sum_{j=1}^{N} \tilde{b}_j p_{jt-1}^{\gamma}}\right), \ \ i = 1, \ldots, N. \qquad \text{(A9)}$$

Rearranging terms in (A9) and recalling that $\pi_t = c(\mathbf{p}_t, \tilde{\mathbf{b}}) / c(\mathbf{p}_{t-1}, \tilde{\mathbf{b}})$ with $c(\cdot)$ defined in (A1), we can solve for $\tilde{b}_i$ as,

$$\tilde{b}_i = \left(\frac{\gamma}{k_1}\right)\left(\frac{\Delta s_{it}}{\Delta \ln s_{it}}\right)\left(\frac{\Delta \ln p_{it} - \ln \pi_t}{p_{it}^{\gamma} - p_{it-1}^{\gamma}\, \pi_t^{\gamma}}\right)\left(\sum_{j=1}^{N} \tilde{b}_j p_{jt}^{\gamma}\right) > 0, \ \ i = 1, \ldots, N. \qquad \text{(A10)}$$

Notice that (A10) determines $\tilde{\mathbf{b}}$ only up to a scalar multiple, so we are free to choose a normalization on $\tilde{\mathbf{b}}$. Specifying this normalization as $\sum_{j=1}^{N} \tilde{b}_j p_{jt}^{\gamma} = 1$, we solve for k_1 by multiplying the right side of (A10) by p_{it}^{γ}, summing over i=1, ..., N, and rearranging terms:

$$k_1 = \gamma \sum_{i=1}^{N}\left(\frac{\Delta s_{it}}{\Delta \ln s_{it}}\right)\left(\frac{\ln \Delta p_{it} - \ln \pi_t}{1 - (p_{it-1}^{\gamma} / p_{it}^{\gamma})\, \pi_t^{\gamma}}\right) > 0. \qquad \text{(A11)}$$

We can substitute (A11) into (A10) to obtain N equations in N unknowns, $\tilde{b}_i$ for i=1,…,N. These equations have the form:

$$\tilde{b}_i = \left(\frac{\Delta s_{it}}{\Delta \ln s_{it}}\right)\left(\frac{\Delta \ln p_{it} - \ln \pi_t}{p_{it}^{\gamma} - p_{it-1}^{\gamma}\,\pi_t^{\gamma}}\right)\left(\sum_{j=1}^{N} \frac{\Delta s_{jt}}{\Delta \ln s_{jt}}\left[\frac{\Delta \ln p_{jt} - \ln \pi_t}{1-(p_{jt-1}^{\gamma}/p_{jt}^{\gamma})\,\pi_t^{\gamma}}\right]\right)^{-1} > 0, \quad i=1, ..., N. \quad \text{(A12)}$$

Recalling that $\pi_t = c(\mathbf{p}_t, \tilde{\mathbf{b}})/c(\mathbf{p}_{t-1}, \tilde{\mathbf{b}})$, these equations are highly nonlinear, but given any arguments $\tilde{\mathbf{b}} > 0$ within π_t on the right of (A12), we determine a solution $\tilde{\mathbf{b}}^* > 0$ on the left. In other words, (A12) provides a continuous mapping $\tilde{\mathbf{b}}^* = F(\tilde{\mathbf{b}})$. Denote the set of parameters $\tilde{\mathbf{b}} \geq 0$ satisfying the normalization $\sum_{i=1}^{N} \tilde{b}_i p_{it}^{\gamma} = 1$ as the simplex S. Choosing $\tilde{\mathbf{b}} \in S$, it is readily verified that $\tilde{\mathbf{b}}^* \in S$, so F is a continuous mapping from S to S, and thus will have a fixed point. Then (A7) holds by construction at this fixed point, so that (A5) follows from (A6).

(iii) Next, we must show that $\tilde{b}_i$ evaluated as in (A10) lies between the bounds described in Proposition 1. The cost shares $s_{i\tau}$ appearing in (A10) are evaluated as in (A2), but using $b_{i\tau}$, with $\tau = t-1, t$. Without loss of generality, we can normalize the price vectors p_τ by a scalar multiple in each period so that $c(\mathbf{p}_\tau, \mathbf{b}_\tau) = 1$, $\tau = t-1, t$. We will drop the normalization on $\tilde{\mathbf{b}}$ that $\sum_{i=1}^{N} \tilde{b}_i p_{it}^{\gamma} = 1$, and instead specify $\sum_{i=1}^{N} \tilde{b}_i p_{it}^{\gamma} = k_2 > 0$. Denoting $B_i \equiv b_{it}/b_{it-1}$, (A10) can then be written as,

$$\tilde{b}_i = b_{it}\left(\frac{k_2}{k_1}\right)\left(\frac{p_{it}^{\gamma} - p_{it-1}^{\gamma} B_i^{-1}}{\gamma \Delta \ln p_{it} + \ln B_i}\right)\left(\frac{\gamma \Delta \ln p_{it} - \gamma \ln \pi_t}{p_{it}^{\gamma} - p_{it-1}^{\gamma}\pi_t^{\gamma}}\right) \qquad \text{(A13a)}$$

$$= \pi_t^{-\gamma} b_{it-1} \left(\frac{k_2}{k_1}\right)\left(\frac{p_{it}^{\gamma} B_i - p_{it-1}^{\gamma}}{\gamma \Delta \ln p_{it} + \ln B_i}\right)\left(\frac{\gamma \Delta \ln p_{it} - \gamma \ln \pi_t}{p_{it}^{\gamma} \pi_t^{-\gamma} - p_{it-1}^{\gamma}}\right). \qquad \text{(A13b)}$$

From concavity of the natural log function we have 1 – (1/z) ≤ ln z ≤ z – 1, for z > 0, and letting $z = B_i(p_{it} / p_{it-1})^{\gamma}$ it follows that,

$$\left(1 - B_i^{-1}\left(\frac{p_{it-1}}{p_{it}}\right)^{\gamma}\right) \leq \left(\ln B_i + \gamma \ln\left(\frac{p_{it}}{p_{it-1}}\right)\right) \leq \left(B_i\left(\frac{p_{it}}{p_{it-1}}\right)^{\gamma} - 1\right). \qquad \text{(A14)}$$

Notice that the last bracketed terms in (A13a, b) are the reciprocals of the previous bracketed terms, but with $B_i = b_{it}/b_{it-1}$ appearing instead of $\pi_t^{-\gamma}$. Suppose that $B_i \geq \pi_t^{-\gamma}$. Using (A14), we can show that:

$$\frac{d}{dB_i}\left(\frac{p_{it}^{\gamma} - p_{it-1}^{\gamma} B_i^{-1}}{\ln B_i + \gamma \Delta \ln p_{it}}\right) \leq 0 \quad \text{and} \quad \frac{d}{dB_i}\left(\frac{p_{it}^{\gamma} B_i - p_{it-1}^{\gamma}}{\ln B_i + \gamma \Delta \ln p_{it}}\right) \geq 0 .$$

It follows by comparing the bracketed terms in (A13) that:

$$\pi_t^{-\gamma} b_{it-1}(k_2 / k_1) \leq \tilde{b}_i \leq b_{it}(k_2 / k_1) , \qquad \text{(A15)}$$

while if $B_i \leq \pi_t^{-\gamma}$ then these inequalities would be reversed. Express π_t from (A5) as:

$$\pi_t = \prod_{i=1}^{N}\left(\frac{p_{it}}{p_{it-1}}\right)^{w_i} = \prod_{i=1}^{N}\left(\frac{b_{it}}{b_{it-1}}\right)^{-w_i/\gamma} \prod_{i=1}^{N}\left(\frac{p_{it} b_{it}^{1/\gamma}}{p_{it-1} b_{it-1}^{1/\gamma}}\right)^{w_i} . \qquad \text{(A16)}$$

A straightforward extension of (A1)-(A4) allowing for $b_{it} \neq b_{it-1}$ shows that the final product in (A16) equals $c(\mathbf{p}_t, \mathbf{b}_t)/c(\mathbf{p}_{t-1},\mathbf{b}_{t-1})$. But this is unity by our normalization of prices, so that $\pi_t^{-\gamma}$ in (A16) equals $\prod_{i=1}^{N}(b_{it}/b_{it-1})^{w_i}$. Then choose k_2 such that $(k_2/k_1) = \left(\prod_{i=1}^{N} b_{it}^{w_i}\right)^{-1}$ Substituting this into (A15), the bounds on $\tilde{\mathbf{b}}$ in Proposition 1 are obtained.

Proof of Proposition 2

Let $\mathbf{b}_t$ denote the vector of the b_{it} from the CES model, let $\beta_{it} = \ln b_{it}$ and let $\boldsymbol{\beta}_t$ denote the vector with components $\ln b_{it}$. Also, to model stochastic tastes, let

$$\beta_{i\tau} = \beta_i^* + e_{i\tau}, \tag{A17}$$

for τ = t or t–1. The $e_{i\tau}$ are assumed to be iid with mean 0 and variance σ_β^2.

Let $\mathbf{w}$ and $\Delta \ln \mathbf{p}_t$ represent vectors of the w_i as in (8) and the log price changes, and let $\mathbf{w}^*$ denote the value of $\mathbf{w}$ when $\boldsymbol{\beta}_{t-1} = \boldsymbol{\beta}_t = \boldsymbol{\beta}^*$. Then,

$$\begin{aligned} \text{var}\,\pi_{sv} &= E[(\Delta \ln \mathbf{p}_t)'(\mathbf{w} - E(\mathbf{w}))(\mathbf{w} - E(\mathbf{w}))'(\Delta \ln \mathbf{p}_t)] \\ &\approx (\Delta \ln \mathbf{p}_t)' \, E[(\mathbf{w} - \mathbf{w}^*)(\mathbf{w} - \mathbf{w}^*)'] \, (\Delta \ln \mathbf{p}_t) \end{aligned} \tag{A18}$$

A linear approximation for $\mathbf{w}$ is:

$$\mathbf{w} \approx \mathbf{w}^* + (\partial \mathbf{w} / \partial \boldsymbol{\beta}_{t-1})\,\mathbf{e}_{t-1} + (\partial \mathbf{w} / \partial \boldsymbol{\beta}_t)\,\mathbf{e}_t \tag{A19}$$

where the derivative matrices are evaluated at the point $\boldsymbol{\beta}_{t-1} = \boldsymbol{\beta}_t = \boldsymbol{\beta}^*$. In Lemma 1 below we show that these derivatives approximately equal:

$$\partial \mathbf{w}_{t-1} / \partial \boldsymbol{\beta}_{t-1} = \partial \mathbf{w}_t / \partial \boldsymbol{\beta}_t \approx \tfrac{1}{2}[\text{diag}(\mathbf{w}^*) - \mathbf{w}^* \, \mathbf{w}^{*\prime}] \tag{A20}$$

where diag($\mathbf{w}^*$) denotes the matrix with the elements $\mathbf{w}^*$ on its main diagonal and zeros elsewhere. Since $E(\mathbf{e}_{t-1}\,\mathbf{e}_t') = 0$, it follows that,

$$E[(\mathbf{w}-\mathbf{w}^*)(\mathbf{w}-\mathbf{w}^*)'] = \tfrac{1}{4}[\text{diag}(\mathbf{w}^*) - \mathbf{w}^*\,\mathbf{w}^{*\prime}]\,[E(\mathbf{e}_t\,\mathbf{e}_t') + E(\mathbf{e}_{t-1}\,\mathbf{e}_{t-1}')][\text{diag}(\mathbf{w}^*) - \mathbf{w}^*\,\mathbf{w}^{*\prime}]$$

$$= \tfrac{1}{2}\sigma_\beta^2\,[\text{diag}(\mathbf{w}^*) - \mathbf{w}^*\,\mathbf{w}^{*\prime}][\text{diag}(\mathbf{w}^*) - \mathbf{w}^*\,\mathbf{w}^{*\prime}]. \qquad \text{(A21)}$$

Then substituting from (A21) into (A18) we obtain:

$$\text{var}\,\pi_{sv} \approx \tfrac{1}{2}\sigma_\beta^2 \sum_{i=1}^{N} w_i^2 (\Delta \ln p_{it} - \pi_{sv})^2 . \qquad \text{(A22)}$$

<u>Lemma 1</u>: *An approximate formula for the derivatives of Sato-Vartia weights with respect to the CES model disturbances is:*

$$\partial \mathbf{w}_t / \partial \boldsymbol{\beta}_{t-1} = \partial \mathbf{w}_t / \partial \boldsymbol{\beta}_t \approx \tfrac{1}{2}[\text{diag}(\mathbf{w}^*) - \mathbf{w}^*\,\mathbf{w}^{*\prime}] \qquad \text{(A20)}$$

Proof:

In section (a) below we show that the elements on the main diagonal of $\partial \mathbf{w}_t / \partial \boldsymbol{\beta}_{t-1}$ are of the form $0.5 w_i (1-w_i)$. Then in section (b) we show that $\partial \mathbf{w}_t / \partial \boldsymbol{\beta}_{t-1}$ has off-diagonal elements of the form $-0.5 w_i w_j$. Furthermore, by symmetry $\partial \mathbf{w}_t / \partial \boldsymbol{\beta}_t$ will have the same form as $\partial \mathbf{w}_t / \partial \boldsymbol{\beta}_{t-1}$.

(i) Solution for $\partial w_i / \partial \beta_{k,t-1}$ for k = i. Denoting the logarithmic mean of the shares by $m_i \equiv (s_{it} - s_{i,t-1}) / \ln(s_{it} / s_{i,t-1})$, or s_{it} if $s_{it} = s_{i,t-1}$, we have:

$$\partial w_i / \partial \beta_{i,t-1} = (\partial w_i / \partial m_i)(\partial m_i / \partial \ln s_{i,t-1})(\partial \ln s_{i,t-1} / \partial \beta_{i,t-1})$$

$$+ \sum_{j \neq i} (\partial w_i / \partial m_j)(\partial m_j / \partial \ln s_{j,t-1})(\partial \ln s_{j,t-1} / \partial \beta_{i,t-1}). \qquad \text{(A23)}$$

In the first term on the right side of (A23),

$$\partial w_i / \partial m_i \;=\; (1 - w_i) / \textstyle\sum_k m_k \qquad \text{(A24)}$$

Also, since $m_i = (s_{it} - s_{i,t-1}) / \ln(s_{it} / s_{i,t-1})$,

$$\partial m_i / \partial \ln s_{i,t-1} = (m_i - s_{i,t-1}) / \ln(s_{i,t} / s_{i,t-1}). \qquad \text{(A25)}$$

Finally, since $s_{i,t-1} = b_{i,t-1}\, p_{it-1}^{1-\eta}\, c_{it-1}^{\eta-1}$ where $c_{it-1} = \left(\sum_{i=1}^{N} b_{it-1} p_{it-1}^{1-\eta}\right)^{1/(1-\eta)}$,

$$\partial \ln s_{i,t-1} / \partial\, \beta_{i,t-1} \;=\; 1 - s_{i,t-1}. \qquad \text{(A27)}$$

Substituting from (A24), (A25) and (A26) into the first term of (A23), we have:

$$(\partial w_i / \partial m_i)(\partial m_i / \partial \ln s_{i,t-1})(\partial \ln s_{i,t-1} / \partial \beta_{i,t-1}) \;=\; \frac{(1 - w_i)(m_i - s_{i,t-1})(1 - s_{i,t-1})}{\ln(s_{i,t} / s_{i,t-1})[\sum_k m_k]}. \qquad \text{(A28)}$$

To find an expression for $(\partial w_i / \partial m_j)(\partial m_j / \partial \ln s_{j,t-1})(\partial \ln s_{j,t-1} / \partial \beta_{i,t-1})$, $j \neq i$, note first that,

$$\partial w_i / \partial m_j \;=\; -w_i / [\textstyle\sum_k m_k]. \qquad \text{(A29)}$$

Also,

$$\partial m_j / \partial \ln s_{j,t-1} = (m_j - s_{j,t-1}) / \ln(s_{j,t} / s_{j,t-1}). \qquad \text{(A30)}$$

And finally,

$$\partial \ln s_{j,t-1} / \partial \beta_{i,t-1} = -s_{i,t-1}. \qquad \text{(A31)}$$

Putting these three factors together gives:

$$\textstyle\sum_{j\neq i} (\partial w_i / \partial m_j)(\partial m_j / \partial \ln s_{j,t-1})(\partial \ln s_{j,t-1} / \partial \beta_{i,t-1}) \;=\; \frac{w_i\, s_{i,t-1}}{\sum_k m_k} \sum_{j\neq i} \frac{m_j - s_{j,t-1}}{\ln(s_{j,t} / s_{j,t-1})}. \qquad \text{(A32)}$$

The two terms in (A23) therefore have a total of,

$$\partial w_i/\partial\beta_{i,t-1} = \frac{(1-w_i)(m_i - s_{i,t-1})(1-s_{i,t-1})}{\ln(s_{i,t}/s_{i,t-1})[\sum_k m_k]} + \frac{w_i\, s_{i,t-1}}{\sum_k m_k}\sum_{j\neq i}\frac{m_j - s_{j,t-1}}{\ln(s_{j,t}/s_{j,t-1})}$$

$$= \frac{(1-w_i)(m_i - s_{i,t-1})}{\ln(s_{i,t}/s_{i,t-1})[\sum_k m_k]} - \frac{s_{i,t-1}(m_i - s_{i,t-1})}{\ln(s_{i,t}/s_{i,t-1})[\sum_k m_k]} + \frac{w_i\, s_{i,t-1}}{\sum_k m_k}\sum_{k}\frac{m_k - s_{k,t-1}}{\ln(s_{k,t}/s_{k,t-1})}. \qquad \text{(A33)}$$

Since m_k approximately equals the midpoint between $s_{k,t-1}$ and $s_{k,t}$,

$$\frac{m_k - s_{k,t-1}}{\ln(s_{k,t}/s_{k,t-1})} \approx 0.5\, m_k. \qquad \text{(A34)}$$

The overall error of approximation in the variance estimate for π_{sv} from substituting from (A34) into (A33) will be inconsequential, both because the individual errors are small and because they are on average zero. Hence:

$$\partial w_i/\partial\beta_{i,t-1} = \frac{(1-w_i)(m_i - s_{i,t-1})}{\ln(s_{i,t}/s_{i,t-1})[\sum_k m_k]} - \frac{s_{i,t-1}(m_i - s_{i,t-1})}{\ln(s_{i,t}/s_{i,t-1})[\sum_k m_k]} + \frac{w_i\, s_{i,t-1}}{\sum_k m_k}\sum_{k}\frac{m_k - s_{k,t-1}}{\ln(s_{k,t}/s_{k,t-1})}$$

$$\approx \frac{0.5(1-w_i)\, m_i}{\sum_k m_k} - \frac{0.5\, s_{i,t-1}\, m_i}{\sum_k m_k} + 0.5\, w_i\, s_{i,t-1} = 0.5(1-w_i)\, w_i\,. \qquad \text{(A35)}$$

(ii) Solution for $\partial w_i/\partial\beta_{k,t-1}$ **for** $k \neq i$**.** A change in $\beta_{k,t-1}$ will effect w_i by changing $s_{i,t-1}$, by changing $s_{k,t-1}$, and by changing any remaining shares:

$$\partial w_i/\partial\beta_{k,t-1} = (\partial w_i/\partial m_i)(\partial m_i/\partial \ln s_{i,t-1})(\partial \ln s_{i,t-1}/\partial\beta_{k,t-1})$$

$$+ (\partial w_i/\partial m_k)(\partial m_j/\partial \ln s_{k,t-1})(\partial \ln s_{k,t-1}/\partial\beta_{k,t-1})$$

$$+ \sum_{j\neq i\cup k}(\partial w_i/\partial m_j)(\partial m_j/\partial \ln s_{j,t-1})(\partial \ln s_{j,t-1}/\partial\beta_{k,t-1}). \qquad \text{(A36)}$$

The components of the first term on the right-side above are:

$$\partial w_i / \partial m_i = (1 - w_i) / [\textstyle\sum_k m_k] \tag{A37}$$

$$\partial m_i / \partial \ln s_{i,t-1} = [m_i - s_{i,t-1}] / \ln(s_{i,t} / s_{i,t-1}) \tag{A38}$$

$$\partial \ln s_{i,t-1} / \partial \beta_{k,t-1} = -s_{k,t-1}. \tag{A39}$$

From (A34), $[m_i - s_{i,t-1}] / [\ln(s_{i,t} / s_{i,t-1}) \{\sum_k m_k\}] \approx 0.5 w_i$. Making this substitution,

$$(\partial w_i / \partial m_i)(\partial m_i / \partial \ln s_{i,t-1})(\partial \ln s_{i,t-1} / \partial \beta_{k,t-1}) \approx -0.5 (1 - w_i) w_i s_{k,t-1}. \tag{A40}$$

Next, decompose $(\partial w_i / \partial m_k)(\partial m_j / \partial \ln s_{k,t-1})(\partial \ln s_{k,t-1} / \partial \beta_{k,t-1})$ as:

$$\partial w_i / \partial m_k = -w_i / [\textstyle\sum_k m_k] \tag{A41}$$

$$\partial m_k / \partial \ln s_{k,t-1} = [m_k - s_{k,t-1}] / \ln(s_{k,t} / s_{k,t-1}) \tag{A42}$$

$$\partial \ln s_{k,t-1} / \partial \beta_{k,t-1} = 1 - s_{k,t-1}. \tag{A43}$$

Last, decompose $\sum_{j \neq i \cup k} (\partial w_i / \partial m_j)(\partial m_j / \partial \ln s_{j,t-1})(\partial \ln s_{j,t-1} / \partial \beta_{k,t-1})$ as,

$$\partial w_i / \partial m_j = -w_i / [\textstyle\sum_k m_k] \tag{A44}$$

$$\partial m_j / \partial \ln s_{j,t-1} = [m_j - s_{j,t-1}] / \ln(s_{j,t} / s_{j,t-1}) \tag{A45}$$

$$\partial \ln s_{j,t-1} / \partial \beta_{k,t-1} = -s_{k,t-1}. \tag{A46}$$

Hence, substituting from (A41) to (A46) and summing the approximations for $(\partial w_i / \partial m_k)(\partial m_j / \partial \ln s_{k,t-1})(\partial \ln s_{k,t-1} / \partial \beta_{k,t-1})$ and $\sum_{j \neq i \cup k} (\partial w_i / \partial m_j)(\partial m_j / \partial \ln s_{j,t-1})(\partial \ln s_{j,t-1} / \partial \beta_{k,t-1})$ gives:

$$\begin{aligned}\textstyle\sum_{j \neq i} (\partial w_i / \partial m_j)(\partial m_j / \partial \ln s_{j,t-1})(\partial \ln s_{j,t-1} / \partial \beta_{k,t-1}) &\approx 0.5 w_i s_{k,t-1} [\textstyle\sum_{j \neq i} w_j] - 0.5 w_i w_k \\ &= 0.5 w_i s_{k,t-1}(1 - w_i) - 0.5 w_i w_k. \end{aligned} \tag{A47}$$

The final step is to combine all the approximations for terms in $\partial w_i / \partial \beta_{k,t-1}$. This gives:

$$\partial w_i / \partial \beta_{k,t-1} = (\partial w_i / \partial m_i)(\partial m_i / \partial \ln s_{i,t-1})(\partial \ln s_{i,t-1} / \partial \beta_{k,t-1}) +$$

$$(\partial w_i / \partial m_k)(\partial m_j / \partial \ln s_{k,t-1})(\partial \ln s_{k,t-1} / \partial \beta_{k,t-1}) + \sum_{j \neq i \cup k} (\partial w_i / \partial m_j)(\partial m_j / \partial \ln s_{j,t-1})(\partial \ln s_{j,t-1} / \partial \beta_{k,t-1})$$

$$\approx -0.5(1-w_i) w_i s_{k,t-1} + 0.5(1-w_i) w_i s_{k,t-1} - 0.5 w_i w_k = -0.5 w_i w_k. \qquad \text{(A48)}$$

Proof of Proposition 3

We prove a more general version of Proposition 3 than that stated in the text. In this version, we suppose that regression (12) is run over goods i=1,…,N and periods t =1,…,T. In addition, we now denote the weights in (8) by w_{it}, and the weighted variance of prices by $s_t^2 = \sum_i w_{it}(\Delta \ln p_{it} - \pi_t)^2$. Finally, let $\bar{\bar{w}}_{\cdot t} \equiv \sum_i w_{it}^2(\Delta \ln p_{it} - \pi_t)^2/s_t^2$ denote the weighted average of the w_{it} that has weights proportional to $w_{it}(\Delta \ln p_{it} - \pi_t)^2$. The more general version (which simplifies to equation (15) when T = 1) is:

Proposition 3′:

Let $\bar{w}_{\cdot t} \equiv \sum w_{it}^2$*, the weighted average of the* w_{it} *that has weights* w_{it}*, let* $\lambda_t \equiv s_t[\sum_\tau s_\tau^2]^{-0.5}$*, and let* $\rho_t \equiv (s_{t-1}s_t)^{-1}[\bar{\bar{w}}_{\cdot t-1}\bar{\bar{w}}_{\cdot t}]^{-0.5}[\sum_i w_{it-1}(\Delta \ln p_{it-1} - \pi_{t-1}) w_{it}(\Delta \ln p_{it} - \pi_t)]$ *denote the autocorrelation of the products* $w_{it}(\Delta \ln p_{it} - \pi_t)$*. Finally, denote the mean squared error of the generalized version of regression (12) by* $s_\varepsilon^2 \equiv \sum_t \sum_i w_{it}\hat{\varepsilon}_{it}^2$*, where* $\hat{\varepsilon}_{it} = \Delta \ln s_{it} - \hat{\delta}_t + (\hat{\eta} - 1)\Delta \ln p_{it}$.

Then an approximately unbiased estimator s_β^2 *for* σ_β^2 *is:*

$$s_\beta^2 = \frac{s_\varepsilon^2}{2[T - \sum_t \overline{w}_{\cdot t} - \sum_t \lambda_t^2 \overline{\overline{w}}_{\cdot t} + 2\sum_{t>1} \lambda_{t-1}\lambda_t (1-\lambda_t^2)\rho_t (\overline{\overline{w}}_{\cdot t-1}\overline{\overline{w}}_{\cdot t})^{0.5}]} . \qquad (15')$$

Formula (15′) *is only approximaely unbiased because it treats the* w_{it} *as predetermined and therefore nonstochastic*.

Proof:

Replacing (1–η) in (12) with γ, let $\hat{\varepsilon}_{it}$ be the fitted value of ε_{it} from:

$$\Delta \ln s_{it} = \delta_t + \gamma \Delta \ln p_{it} + \varepsilon_{it} , \qquad (12)$$

and let $s_\varepsilon^2 \equiv \sum_t \sum_i w_{it} \hat{\varepsilon}_{it}^2$, the weighted sum of squared errors of the regression equation (12). Substituting from equation (A17) into equation (14) implies that $\varepsilon_{it} = e_{it} - e_{i,t-1} - \sum_j w_{jt}(e_{jt} - e_{j,t-1})$, where $e_{i,t-1}$ and e_{it} have variance σ_β^2. Furthermore, $\sum_i w_{it}[\delta_t + \sum_j w_{jt}(e_{jt} - e_{j,t-1})](\Delta\ln p_{it} - \pi_t) = [\delta_t + \sum_j w_{jt}(e_{jt} - e_{j,t-1})][\sum_i w_{it}(\Delta\ln p_{it} - \pi_t)] = 0$, so it follows that $\sum_i w_{it}(\Delta\ln s_{it})(\Delta\ln p_{it} - \pi_t) = \gamma[\sum_i w_{it}(\Delta\ln p_{it} - \pi_t)^2] + \sum_i w_{it}(e_{it} - e_{i,t-1})(\Delta\ln p_{it} - \pi_t)$. Consequently, a weighted least squares estimator of γ in equation (12) is:

$$\hat{\gamma} = \frac{\sum_t \sum_i w_{it} (\Delta\ln s_{it})(\Delta\ln p_{it} - \pi_t)}{\sum_t \sum_i w_{it} (\Delta\ln p_{it} - \pi_t)^2}$$

$$= \gamma + \frac{\sum_t \sum_i w_{it}(e_{it} - e_{i,t-1})(\Delta\ln p_{it} - \pi_t)}{\sum_t \sum_i w_{it} (\Delta\ln p_{it} - \pi_t)^2}$$

$$= \gamma + \frac{\sum_t \sum_i \lambda_t w_{it} \tilde{p}_{it}(e_{it} - e_{i,t-1})}{[\sum_t s_t^2]^{0.5}} \qquad (A49)$$

where $s_t^2 \equiv \sum_i w_{it}(\Delta\ln p_{it} - \pi_t)^2$, $\lambda_t \equiv s_t/[\sum_\tau s_\tau^2]^{0.5}$, and $\tilde{p}_{it} \equiv (\Delta\ln p_{it} - \pi_t)/s_t$. The i[th] regression error is:

$$\hat{\varepsilon}_{it} = \Delta\ln s_{it} - \hat{\gamma}(\Delta\ln p_{it} - \pi_t)$$

$$= e_{it} - e_{i,t-1} - \sum_j w_j(e_{jt} - e_{j,t-1}) - \lambda_t \tilde{p}_{it}\{\sum_\tau \lambda_\tau [\sum_j w_{j\tau} \tilde{p}_{j\tau}(e_{j\tau} - e_{j\tau-1})]\}$$

$$= [1 - w_{it}(1 + \lambda_t^2 \tilde{p}_{it}^2)](e_{it} - e_{i,t-1}) - \sum_{j\neq i} w_{jt}(1 + \lambda_t^2 \tilde{p}_{it} \tilde{p}_{jt})(e_{jt} - e_{j,t-1})$$

$$- \lambda_t \tilde{p}_{it}\{\sum_{\tau\neq t} \lambda_\tau [\sum_j w_{j\tau} \tilde{p}_{j\tau}(e_{j\tau} - e_{j,\tau-1})]\}. \quad \text{(A50)}$$

Since the $e_{i,t-1}$ and the e_{it} have been assumed to be independent of one another, $E[(e_{it} - e_{i,t-1})^2] = 2\sigma_\beta^2$. Also, $E[(e_{it} - e_{i,t-1})(e_{i\tau} - e_{i,\tau-1})] = -\sigma_\beta^2$ if $\tau = t+1$ or $t-1$, but all other covariances equal zero. For example, in case when t=1, we have:

$$E[\hat{\varepsilon}_{i1}^2]/2\sigma_\beta^2 = (1 - w_{i1}(1 + \lambda_1^2 \tilde{p}_{i1}^2))^2 + \sum_{j\neq i} w_{j1}^2(1 + \lambda_1^2 \tilde{p}_{i1} \tilde{p}_{j1})^2 + \lambda_1^2 \tilde{p}_{i1}^2 \{\sum_{\tau>1} \lambda_\tau^2 [\sum_j w_{j\tau}^2 \tilde{p}_{j\tau}^2]\}$$

$$+ [1 - w_{i1}(1 + \lambda_1^2 \tilde{p}_{i1}^2)]\lambda_1\lambda_2 w_{i2} \tilde{p}_{i1}\tilde{p}_{i2} - \sum_{j\neq i} w_{j1}(1 + \lambda_1^2 \tilde{p}_{i1}\tilde{p}_{j1})\lambda_1\lambda_2 w_{j2} \tilde{p}_{i1}\tilde{p}_{j2}$$

$$- \lambda_1 \tilde{p}_{i1}\{\sum_{\tau=3,\ldots,T} \lambda_\tau \lambda_{\tau-1}[\sum_j (w_{j\tau} \tilde{p}_{j\tau})(w_{j\tau-1} \tilde{p}_{j\tau-1})]\}$$

$$= 1 - 2w_{i1}(1 + \lambda_1^2 \tilde{p}_{i1}^2) + \sum_{j=1,\ldots,N} w_{j1}^2(1 + \lambda_1^2 \tilde{p}_{i1} \tilde{p}_{j1})^2 + \lambda_1^2 \tilde{p}_{i1}^2 \{\sum_{\tau>1} \lambda_\tau^2 [\sum_j w_{j\tau}^2 \tilde{p}_{j\tau}^2]\}$$

$$+ \lambda_1\lambda_2 w_{i2} \tilde{p}_{i1}\tilde{p}_{i2} - \tilde{p}_{i1}\lambda_1\lambda_2 [\sum_j w_{j1}(1 + \lambda_1^2 \tilde{p}_{i1}\tilde{p}_{j1})w_{j2} \tilde{p}_{j2}]$$

$$- \lambda_1 \tilde{p}_{i1}\{\sum_{\tau=3,\ldots,T} \lambda_\tau \lambda_{\tau-1}[\sum_j (w_{j\tau} \tilde{p}_{j\tau})(w_{j\tau-1} \tilde{p}_{j\tau-1})]\}$$

$$= 1 - 2w_{i1} - 2w_{i1}\lambda_1^2 \tilde{p}_{i1}^2 + \sum_j w_{j1}^2 + 2\lambda_1^2 \tilde{p}_{i1}[\sum_j w_{j1}^2 \tilde{p}_{j1}]$$

$$+ \lambda_1^2 \tilde{p}_{i1}^2 \{\sum_{\tau=1,\ldots,T} \lambda_\tau^2 [\sum_j w_{j\tau}^2 \tilde{p}_{j\tau}^2]\} + \lambda_1\lambda_2 \tilde{p}_{i1} w_{i2} \tilde{p}_{i2}$$

$$- \tilde{p}_{i1}\lambda_1\lambda_2[\sum_j w_{j1} w_{j2} \tilde{p}_{j2}] - \tilde{p}_{i1}^2 \lambda_1^3 \lambda_2 [\sum_j w_{j1} \tilde{p}_{j1} w_{j2} \tilde{p}_{j2}]$$

$$- \lambda_1 \tilde{p}_{i1}\{\sum_{\tau=3,\ldots,T} \lambda_\tau \lambda_{\tau-1}[\sum_j (w_{j\tau} \tilde{p}_{j\tau})(w_{j\tau-1} \tilde{p}_{j\tau-1})]\}. \quad \text{(A51)}$$

To express the weighted mean of the expected squares of the time period 1 regression errors in a convenient way, let $\bar{w}_{\cdot t}$ denote $\sum_i w_{it}^2$, and let $\bar{\bar{w}}_{\cdot t}$ denote a weighted average of the w_{it} that has weights proportional to $w_{it}\tilde{p}_{it}^2$. (That is, $\bar{\bar{w}}_{\cdot t} = \sum_i w_{it}^2(\Delta\ln p_{it} - \pi_t)^2 / \sum_i w_{it}(\Delta\ln p_{it} - \pi_t)^2$.) Furthermore, define ρ_t as the (unweighted) correlation between $w_{jt}\,\tilde{p}_{jt}$ and $w_{j,t-1}\,\tilde{p}_{j,t-1}$. We use the following equalities to make substitutions:

$$\sum_i w_{it} = 1\,, \qquad \sum_i w_{it}\tilde{p}_{it}^2 = 1\,,$$

$$\sum_i w_{it}^2 \equiv \bar{w}_{\cdot t}\,, \qquad \sum_i w_{it}^2\,\tilde{p}_{it}^2 \equiv \bar{\bar{w}}_{\cdot t}\,,$$

$$\sum_i w_{it}\tilde{p}_{it} = 0\,, \qquad \sum_i w_{i,t-1}\tilde{p}_{i,t-1}w_{it}\tilde{p}_{it} = \rho_t[\bar{\bar{w}}_{\cdot t-1}\bar{\bar{w}}_{\cdot t}]^{0.5}.$$

These substitutions give the result:

$$[\sum w_{i1}E(\hat{\varepsilon}_{i1}^2)]/2\sigma_\beta^2 = 1 - \bar{w}_{\cdot 1} - 2\lambda_1^2\,\bar{\bar{w}}_{\cdot 1} + \lambda_1^2\,[\sum_{\tau=1,\ldots,T}\ \lambda_\tau^2\,\bar{\bar{w}}_{\cdot\tau}]$$

$$+ [\lambda_1\lambda_2 - \lambda_1^3\lambda_2][\sum_j w_{j1}\,\tilde{p}_{j1}w_{j2}\,\tilde{p}_{j2}]$$

$$= 1 - \bar{w}_{\cdot 1} - 2\lambda_1^2\,\bar{\bar{w}}_{\cdot 1} + \lambda_1^2\,[\sum_{\tau=1,\ldots,T}\ \lambda_\tau^2\,\bar{\bar{w}}_{\cdot\tau}] + \lambda_1(1-\lambda_1^2)\lambda_2\,\rho_2\,[\bar{\bar{w}}_{\cdot 1}\bar{\bar{w}}_{\cdot 2}]^{0.5} \qquad \text{(A52)}$$

When $1 < t < T$, the expression for $E(\hat{\varepsilon}_{it}^2)$ contains autocorrelations between both periods t–1 and t and periods t+1 and t. Its derivation is as follows:

$$E[\hat{\varepsilon}_{it}^2]/2\sigma_\beta^2 = (1 - w_{it}(1 + \lambda_t^2\,\tilde{p}_{it}^2))^2 + \sum_{j\neq i} w_{jt}^2(1 + \lambda_t^2\,\tilde{p}_{it}\,\tilde{p}_{jt})^2$$

$$+ \lambda_t^2\,\tilde{p}_{it}^2\{\sum_{\tau\neq t}\lambda_\tau^2\,[\sum_j w_{j\tau}^2\,\tilde{p}_{j\tau}^2]\}$$

$$+ [1 - w_{it}(1 + \lambda_t^2\,\tilde{p}_{it}^2)]\lambda_t\tilde{p}_{it}[\lambda_{t-1}w_{i,t-1}\tilde{p}_{i,t-1} + \lambda_{t+1}w_{i,t+1}\tilde{p}_{i,t+1}]$$

$$- \sum_{j\neq i} w_{jt}(1 + \lambda_t^2\,\tilde{p}_{it}\tilde{p}_{jt})\lambda_t\tilde{p}_{it}[\lambda_{t-1}w_{j,t-1}\tilde{p}_{j,t-1} + \lambda_{t+1}w_{j,t+1}\tilde{p}_{j,t+1}]$$

$$- \lambda_t\,\tilde{p}_{it}\{\sum_{\tau\neq 1\cup t\cup t+1}\lambda_\tau\,[\sum_j (w_{j\tau}\,\tilde{p}_{j\tau})(w_{j,\tau-1}\,\tilde{p}_{j,\tau-1})]\}$$

$$= 1 - 2w_{it} - 2w_{it}\lambda_t^2\,\tilde{p}_{it}^2 + \bar{w}_{\cdot t} + 2\lambda_t^2\,\tilde{p}_{it}[\textstyle\sum_j w_{jt}^2\,\tilde{p}_{jt}]$$
$$+ \lambda_t^2\,\tilde{p}_{it}^2\,\{\textstyle\sum_\tau \lambda_\tau^2\,[\sum_j w_{j\tau}^2\,\tilde{p}_{j\tau}^2]\}$$
$$+ \lambda_t\tilde{p}_{it}[\lambda_{t-1}w_{i,t-1}\tilde{p}_{i,t-1} + \lambda_{t+1}w_{i,t+1}\tilde{p}_{i,t+1}]$$
$$- \lambda_t^3\,\tilde{p}_{it}^2\{\textstyle\sum_j w_{jt}\,\tilde{p}_{jt}\,[\lambda_{t-1}w_{j,t-1}\tilde{p}_{j,t-1} + \lambda_{t+1}w_{j,t+1}\tilde{p}_{j,t+1}]\}$$
$$- \lambda_t\,\tilde{p}_{it}\{\textstyle\sum_{\tau\neq 1\cup t\cup t+1} \lambda_\tau\,[\sum_j (w_{j\tau}\,\tilde{p}_{j\tau})(w_{j,\tau-1}\,\tilde{p}_{j,\tau-1})]\}. \qquad \text{(A53)}$$

Consequently, for $1 < t < T$,

$$E[\textstyle\sum_i w_{it}\,\hat{\varepsilon}_{it}^2]/2\sigma_\beta^2 = 1 - \bar{w}_{\cdot t} - 2\lambda_t^2\,\bar{\bar{w}}_{\cdot t} + \lambda_t^2[\sum_\tau \lambda_\tau^2\,\bar{\bar{w}}_{\cdot\tau}] + \lambda_{t-1}\lambda_t\,(1-\lambda_t^2)\rho_t[\bar{\bar{w}}_{\cdot t-1}\,\bar{\bar{w}}_{\cdot t}]^{0.5}$$
$$+ \lambda_t(1-\lambda_t^2)\lambda_{t+1}\rho_{t+1}[\bar{\bar{w}}_{\cdot t}\,\bar{\bar{w}}_{\cdot t+1}]^{0.5}. \qquad \text{(A54)}$$

Using the fact that $\sum\lambda_t^2 = 1$, the sum over all time periods is:

$$E[\textstyle\sum_t \sum_i w_{it}\hat{\varepsilon}_{it}^2]/2\sigma_\beta^2 = T - \sum\bar{w}_{\cdot t} - \sum\lambda_t^2\,\bar{\bar{w}}_{\cdot t}$$
$$+ 2\textstyle\sum_{t=2,\ldots,T} \lambda_{t-1}\lambda_t\,(1-\lambda_t^2)\rho_t[\bar{\bar{w}}_{\cdot t-1}\,\bar{\bar{w}}_{\cdot t}]^{0.5}. \qquad \text{(A55)}$$

The sum of squared errors $\sum_t \sum_i w_{it}\hat{\varepsilon}_{it}^2$ divided by 2 times the right side of (A55) therefore has an expected value of σ_β^2, which is the result in Proposition 3′.

<u>Proof of Proposition 4</u>

In this section we denote the vector of log price changes $\Delta\ln\mathbf{p}_t$ by $\dot{\mathbf{p}}$, and assume that these have a random error term of $\mathbf{e}^p$, which may be heteroskedastic. That is,

$$\dot{\mathbf{p}} = \mu^p + \mathbf{e}^p. \qquad \text{(A56)}$$

Denote the variance of $\dot{p}_i$ by σ_i^2 and denote the estimate of this variance by s_i^2. Although an assumption that $\dot{p}_i$ has a positive covariance with w_i is appealing because positive shocks to b_i raise equilibrium market prices, it adds excessive complexity to the expression for var π. (See Mood, Graybill and Boes, 1974, p. 180.) Hence, for the sake of simplicity, we will assume that the shocks to prices are independent of the error term for preferences. We continue to assume that the taste parameters are distributed as in (A17), now written as $\beta_\tau = \beta^* + \mathbf{e}_\tau$, for τ = t-1,t.

To obtain an estimator for the variance, we use the following linear approximation for $\mathbf{w}$:

$$\begin{aligned} \mathbf{w} &\approx \mu^w + (\partial \mathbf{w}/\partial \beta_{t-1})\mathbf{e}_{t-1} + (\partial \mathbf{w}/\partial \beta_t)\mathbf{e}_t \\ &\approx \mu^w + \mathbf{G}\mathbf{e}^w \end{aligned} \tag{A57}$$

where μ^w denotes the value of $\mathbf{w}$ when $\beta_{t-1} = \beta_t = \beta^*$, $\partial \mathbf{w}/\partial \beta_{t-1}$ and $\partial \mathbf{w}/\partial \beta_t$ are evaluated at $\beta_{t-1} = \beta_t = \beta^*$ and are estimated by $\mathbf{G} = 0.5[\text{diag}(\mathbf{w}) - \mathbf{w}\mathbf{w}']$, and where $\mathbf{e}^w = \mathbf{e}_{t-1} + \mathbf{e}_t$. We then have,

$$\pi_{sv} = \mathbf{w}'\dot{\mathbf{p}} \approx (\mu^w + \mathbf{G}\mathbf{e}^w)'(\mu^p + \mathbf{e}^p). \tag{A58}$$

Using the independence assumption to eliminate the expected values of cross-products of error terms, we have:

$$E[\pi_{sv}^2 - [E(\pi_{sv})]^2] \approx E(\mu^{w\prime}\mathbf{e}^p)^2 + E(\mathbf{e}^{w\prime}\mathbf{G}'\mu^p)^2 + E(\mathbf{e}^{w\prime}\mathbf{G}'\mathbf{e}^p)^2. \tag{A59}$$

To obtain an estimator for the first term on the right side of (A59), substitute $\mathbf{w}$ for μ^w and s_i^2 for σ_i^2:

$$E(\mu^{w\prime}\mathbf{e}^p)^2 = \sum_i (\mu_i^w)^2 \sigma_i^2 \approx \sum_i w_i^2 s_i^2. \tag{A60}$$

Similarly, to obtain an estimator for $E(\mathbf{e}^{w\prime}\mathbf{G}'\mu^p)^2$, substitute $\dot{\mathbf{p}}$ for μ^p and s_β^2 for σ_β^2:

$$E(\mathbf{e}^{w\prime}\mathbf{G}'\boldsymbol{\mu}^p)^2 \;\approx\; \tfrac{1}{2}\sigma_\beta^2 \textstyle\sum_i w_i^2[\mu_i^p - (\mathbf{w}'\boldsymbol{\mu}^p)]^2 \approx \tfrac{1}{2}s_\beta^2 \textstyle\sum_i w_i^2(\dot{p}_i - \pi_{sv})^2. \qquad \text{(A61)}$$

To estimate the third term, note that $E(\mathbf{e}^{w\prime}\mathbf{G}'\mathbf{e}^p)^2 = E(\mathbf{e}^{p\prime}\mathbf{G}\mathbf{e}^w\mathbf{e}^{w\prime}\mathbf{G}'\mathbf{e}^p) = 2\sigma_\beta^2 E(\mathbf{e}^{p\prime}\mathbf{G}\mathbf{G}'\mathbf{e}^p)$. Letting $\mathbf{g}$ denote a vector equal to the main diagonal of $\mathbf{GG}'$ and using the independence assumption to set the expected value of the cross-product terms equal to zero gives,

$$E(\mathbf{e}^{p\prime}\mathbf{G}\mathbf{G}'\mathbf{e}^p) \;=\; [\sigma_1^2 \;\; \sigma_2^2 \;\; \ldots \;\; \sigma_n^2]\,\mathbf{g} \qquad \text{(A62)}$$

Letting $\mathbf{w}^2$ denote the vector of the w_i^2 and noting that $\mathbf{w}'\mathbf{w} = \overline{\overline{w}}$, $\mathbf{G}\mathbf{G}'$ equals $\frac{1}{4}[\text{diag}(\mathbf{w}^2) - (\mathbf{w}^2)\mathbf{w}'$ $- \mathbf{w}(\mathbf{w}^2)' + \overline{\overline{w}}(\mathbf{w}\mathbf{w}')]$. Hence, the i[th] element of $\mathbf{g}$, equals $\frac{1}{4}w_i^2(1 + \overline{\overline{w}} - 2w_i)$ and,

$$E(\mathbf{e}^{p\prime}\mathbf{G}\mathbf{G}'\mathbf{e}^p) \;=\; \tfrac{1}{4}\textstyle\sum_i \sigma_i^2 w_i^2(1 + \overline{\overline{w}} - 2w_i). \qquad \text{(A63)}$$

Therefore,

$$E(\mathbf{e}^{p\prime}\mathbf{G}\mathbf{e}^w\mathbf{e}^{w\prime}\mathbf{G}'\mathbf{e}^p) \;=\; \tfrac{1}{2}\sigma_\beta^2 \textstyle\sum_i \sigma_i^2 w_i^2(1 + \overline{\overline{w}} - 2w_i). \qquad \text{(A64)}$$

Finally, adding together the estimators for the three terms gives:

$$E[\pi_{sv}^2 - [E(\pi_{sv})]^2] \approx \textstyle\sum (\mu_i^w)^2\sigma_i^2 + \tfrac{1}{2}\sigma_\beta^2 \sum w_i^2(\dot{p}_i - \pi)^2 + \tfrac{1}{2}\sigma_\beta^2 \sum \sigma_i^2 w_i^2(1 + \overline{\overline{w}} - 2w_i). \qquad \text{(A65)}$$

We can estimate this expression by:

$$\text{var}\,\pi_{sv} \approx \textstyle\sum w_i^2 s_i^2 + \tfrac{1}{2}s_\beta^2 \sum w_i^2(\dot{p}_i - \pi)^2 + \tfrac{1}{2}s_\beta^2 \sum s_i^2 w_i^2(1 + \overline{\overline{w}} - 2w_i),$$

$$= \textstyle\sum w_i^2 s_i^2 + \dfrac{s_\varepsilon^2 s_p^2 \overline{\overline{w}}}{4(1 - \overline{w} - \overline{\overline{w}})} + \dfrac{s_\varepsilon^2 \sum w_i^2 s_i^2 (1 + \overline{w} - 2w_i)}{4(1 - \overline{w} - \overline{\overline{w}})}. \qquad \text{(A66)}$$

where the expression subsituted for s_β^2 comes from Proposition 3.

Note that equation (A66) is only an approximately unbiased, because the responses of demand to price disturbances add a (presumably small) to additional component to the variances of the w_i that is ignored in (A66). Note also that in the special case where all prices have the same distribution, so that $s_i^2 = s_p^2/(1-\overline{w})$, the variance of the index may be estimated as:

$$\operatorname{var}\pi_{sv} \approx \frac{s_p^2\overline{w}}{(1-\overline{w})} + \frac{s_\varepsilon^2 s_p^2 \overline{\overline{w}}}{4(1-\overline{w}-\overline{\overline{w}})} + \frac{s_\varepsilon^2 s_p^2 \overline{w}(1+\overline{w})}{4(1-\overline{w}-\overline{\overline{w}})(1-\overline{w})} - \frac{s_\varepsilon^2 s_p^2 \overline{w}\sum w_i^3}{2(1-\overline{w}-\overline{\overline{w}})(1-\overline{w})}. \qquad \text{(A67)}$$

Proof of Proposition 5

Taking the log of (22), we obtain,

$$\ln c(p_t,\overline{\alpha}) - \ln c(p_{t-1},\overline{\alpha})$$

$$= \frac{1}{2}\left[\sum_{i=1}^{N}(\alpha_{it}+\alpha_{it-1})\ln(p_{it}/p_{it-1}) + \sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{ij}\ln p_{it}\ln p_{jt} - \sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{ij}\ln p_{it-1}\ln p_{jt-1}\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{N}(\alpha_{it}+\alpha_{it-1})\ln(p_{it}/p_{it-1}) + \sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{ij}(\ln p_{it}+\ln p_{it-1})(\ln p_{jt}-\ln p_{jt-1})\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{N}(s_{it}+s_{it-1})\ln(p_{it}/p_{it-1})\right], \qquad \text{(A68)}$$

where the second line follows by using the translog formula in (19), the third line using simple algebra, and the final line follows from the share formula in (20).

Proof of Proposition 6

Proved in the main text.

Proof of Proposition 7

We now assume that ln $\mathbf{p}_t = \mathbf{p}^*_t + \mathbf{u}_t$, where $\mathbf{u}_t$ is an error term with $E(u_{it}) = 0$ and u_{it} independent of u_{jt} and $u_{i,t-1}$. Define $\mu^p{}_t$ as $\mathbf{p}^*_t - \ln \mathbf{p}^*_{t-1}$ and $\mathbf{v}_t$ as $\mathbf{u}_t - \mathbf{u}_{t-1}$. We assume that the i^{th} element of $\mathbf{v}_t$ has variance σ_i^2. Then, letting $\boldsymbol{\Gamma}$ represent the matrix of the γ_{ij}, from equation (21),

$$\mathbf{w} = \alpha + \tfrac{1}{2}\boldsymbol{\Gamma}(\ln \mathbf{p}_t + \ln \mathbf{p}_{t-1}) + \tfrac{1}{2}(\boldsymbol{\varepsilon}_t + \boldsymbol{\varepsilon}_{t-1})$$

$$= \alpha + \tfrac{1}{2}\boldsymbol{\Gamma}(\mu^p{}_t + \mu^p{}_{t-1} + \mathbf{u}_t + \mathbf{u}_{t-1}) + \tfrac{1}{2}(\boldsymbol{\varepsilon}_t + \boldsymbol{\varepsilon}_{t-1})$$

$$= \mu^w{}_t + \mathbf{e}_t \qquad \text{(A69)}$$

where $\mathbf{e}_t = \tfrac{1}{2}[\boldsymbol{\Gamma}(\mathbf{u}_t + \mathbf{u}_{t-1}) + (\boldsymbol{\varepsilon}_t + \boldsymbol{\varepsilon}_{t-1})]$, $E(\boldsymbol{\varepsilon}_t\,\boldsymbol{\varepsilon}_t') = \boldsymbol{\Omega}$, and $E(\boldsymbol{\varepsilon}_t\,\boldsymbol{\varepsilon}_{t-1}') = \rho\,\boldsymbol{\Omega}$. Let $\boldsymbol{\Sigma}$ denote $E[(\mathbf{u}_t + \mathbf{u}_{t-1})(\mathbf{u}_t + \mathbf{u}_{t-1})']$,where the main diagonal of $\boldsymbol{\Sigma}$ equals the σ_i^2 and its off-diagonal elements equal 0 because the u_{it} are assumed to be independently distributed. We also assume that $(\mathbf{u}_t + \mathbf{u}_{t-1})$ is independent of $\boldsymbol{\varepsilon}_t$ and $\boldsymbol{\varepsilon}_{t-1}$. Hence, $E(\mathbf{e}_t\,\mathbf{e}_t') = \tfrac{1}{4}\boldsymbol{\Gamma}\boldsymbol{\Sigma}\boldsymbol{\Gamma}' + \tfrac{1}{2}(1+\rho)\boldsymbol{\Omega}$.

The Törnqvist index, which we denote by π, may be written as:

$$\pi = (\mu^w + \mathbf{e}_t)'(\mu^p{}_t + \mathbf{v}_t).$$

$$= \mu^{w\prime}\mu^p{}_t + \mu^{w\prime}\mathbf{v}_t + \mu^{p\prime}_t\,\mathbf{e}_t + \mathbf{e}_t'\,\mathbf{v}_t. \qquad \text{(A70)}$$

Note that $E[(\mathbf{u}_t - \mathbf{u}_{t-1})(\boldsymbol{\varepsilon}_t + \boldsymbol{\varepsilon}_{t-1})'] = 0$ and that $E(\mathbf{e}_t'\,\mathbf{v}_t) = 0$. ($E(\mathbf{e}_t'\,\mathbf{v}_t) = E[(\mathbf{u}_t + \mathbf{u}_{t-1})\boldsymbol{\Gamma}' + (\boldsymbol{\varepsilon}_t + \boldsymbol{\varepsilon}_{t-1})'](\mathbf{u}_t - \mathbf{u}_{t-1})$, which equals 0 since $E[\gamma_{ij}(u_{it} + u_{i,t-1})(u_{jt} - u_{j,t-1})] = 0$ for $i\neq j$ and $\gamma_{ii}^2\,E(u_{it}^2 - u_{i,t-1}^2) = 0$.) It follows that,

$$E(\pi) = \mu^{w\prime}\mu^p{}_t. \qquad \text{(A71)}$$

In addition, because the cross-products of the terms in (A70) have expected values of 0, we have

$$E(\pi^2) - [E(\pi)]^2 = E(\mu^{w\prime}\mathbf{v}_t\,\mathbf{v}_t'\,\mu^w) + E(\mu^{p\prime}_t\,\mathbf{e}_t\,\mathbf{e}_t'\,\mu^p{}_t) + E[(\mathbf{e}_t'\,\mathbf{v}_t)^2]. \qquad \text{(A72)}$$

We can substitute the following expressions for the terms in (A72) :

$$E(\mu^{w\prime}\mathbf{v}_t\mathbf{v}_t^{\prime}\,\mu^{w}) = \sum (\mu^{w}_{it})^2\,\sigma_i^{\,2} \qquad \text{(A73)}$$

$$E[\mu^{p}{}_t^{\prime}\,\mathbf{e}_t\,\mathbf{e}_t^{\prime}\,\mu^{p}{}_t] \;=\; \tfrac{1}{4}\mu^{p}{}_t^{\prime}\,\boldsymbol{\Gamma\Sigma\Gamma}^{\prime}\,\mu^{p}{}_t \;+\; \tfrac{1}{2}\mu^{p}{}_t^{\prime}(1+\rho)\,\boldsymbol{\Omega}\,\mu^{p}{}_t \qquad \text{(A74)}$$

To evaluate $E[(\mathbf{e}_t^{\prime}\,\mathbf{v}_t)^2]$, note that $E(e_{it}v_{it}e_{jt}v_{jt}) = 0$ because v_{it} is independent of the other terms in this product. Also,

$$E[(e_{it}v_{it})^2] = E\Big\{\big[\tfrac{1}{2}[\textstyle\sum_j \gamma_{ij}(u_{jt} + u_{j,t-1})](u_{it} - u_{i,t-1}) + \tfrac{1}{2}(\varepsilon_{it} + \varepsilon_{i,t-1})(u_{it} - u_{i,t-1})\big]^2\Big\}$$

$$= \tfrac{1}{4}\sigma_i^{\,2}[\textstyle\sum_j \gamma_{ij}^{\,2}\,\sigma_j^{\,2}] \;+ \tfrac{1}{2}(\sigma_i^{\,2})(1+\rho)\Omega_{ii}\,.$$

Hence =

$$E[(\mathbf{e}_t^{\prime}\,\mathbf{v}_t)^2] \;=\; \tfrac{1}{4}\textstyle\sum_i \sigma_i^{\,2}\,[\sum_j \gamma_{ij}^{\,2}\,\sigma_j^{\,2}] \;+\; \tfrac{1}{2}\sum_i (\sigma_i^{\,2})(1+\rho)\Omega_{ii} \qquad \text{(A75)}$$

where Ω_{ii} represents the elements on the main diagonal of $\boldsymbol{\Omega}$. Substituting (A73)- (A75) into (A72), we have:

$$\mathrm{var}(\pi) = \sum (\mu^{w}_{it})^2\,\sigma_i^{\,2} \;+\; \tfrac{1}{4}\mu^{p}{}_t^{\prime}\,\boldsymbol{\Gamma}^{\prime}\boldsymbol{\Sigma\Gamma}\mu^{p}{}_t \;+\; \tfrac{1}{2}(1+\rho)\,\mu^{p}{}_t^{\prime}\boldsymbol{\Omega}\,\mu^{p}{}_t$$

$$+\; \tfrac{1}{4}\textstyle\sum_i \sigma_i^{\,2}\,[\sum_j \gamma_{ij}^{\,2}\,\sigma_j^{\,2}] \;+\; \tfrac{1}{2}(1+\rho)[\sum \Omega_{ii}\,\sigma_i^{\,2}]\,. \qquad \text{(A76)}$$

To estimate var(π) using the expression in (A76), estimate μ^{w}_{it} by w_{it}, estimate $\sigma_i^{\,2}$ by $s_i^{\,2}$, where $s_i^{\,2}$ may be the sample variance for the log changes in the individual prices collected for item i. In addition, μ^{p}_{it} can be estimated by $\Delta\ln p_{it}$, and $\boldsymbol{\Gamma}$, ρ and $\boldsymbol{\Omega}$ can be estimated from the multi-period version of regression (21), which may take the form of regression (25). Substituting $\sum_i \sum_j (\Delta\ln p_{it})(\Delta\ln p_{jt})\,[\sum_k \gamma_{ik}\gamma_{jk}\,s_k^{\,2}]$ for $\mu^{p}{}_t^{\prime}\,\boldsymbol{\Gamma}^{\prime}\boldsymbol{\Sigma\Gamma}\mu^{p}{}_t$, an estimator for var($\pi$) is, then:

$$\mathrm{var}(\pi) \;\approx\; \sum w_{it}^{\,2}\,s_i^{\,2} \;+\tfrac{1}{4}\textstyle\sum_i \sum_j (\Delta\ln p_{it})(\Delta\ln p_{jt})\,[\sum_k \gamma_{ik}\gamma_{jk}\,s_k^{\,2}]$$

$$+ \tfrac{1}{2}(1+\hat{\rho})(\Delta \ln \mathbf{p}_t)' \hat{\boldsymbol{\Omega}} (\Delta \ln \mathbf{p}_t)$$

$$+ \tfrac{1}{4} \textstyle\sum_i (s_i^{\,2})[\sum_j \hat{\gamma}_{ij}^{\,2} s_j^{\,2}] + \tfrac{1}{2}(1+\hat{\rho})[\sum \hat{\Omega}_{ii}\, s_i^{\,2}]. \qquad \text{(A77)}$$

In the case where all prices have the same trend and variance, we can estimate $\sigma_i^{\,2}$ by $s_p^2/(1-\overline{w})$ for all i. Then (A77) becomes:

$$\mathrm{var}(\pi) \approx s_p^2\, \overline{w}/(1-\overline{w}) + \tfrac{1}{4}[s_p^2/(1-\overline{w})]\,(\Delta \ln \mathbf{p}_t)' \hat{\Gamma} \hat{\Gamma} (\Delta \ln \mathbf{p}_t) + \tfrac{1}{2}(1+\hat{\rho})(\Delta \ln \mathbf{p}_t)' \hat{\boldsymbol{\Omega}} (\Delta \ln \mathbf{p}_t)$$

$$+ \tfrac{1}{4}[s_p^2/(1-\overline{w})]^2 [\textstyle\sum_i \sum_j \hat{\gamma}_{ij}^{\,2}] + \tfrac{1}{2}[s_p^2/(1-\overline{w})](1+\hat{\rho})[\sum \hat{\Omega}_{ii}] \qquad \text{(A77}'\text{)}$$

where $(\Delta \ln \mathbf{p}_t)' \hat{\Gamma} \hat{\Gamma} (\Delta \ln \mathbf{p}_t)$ can be expressing as $\sum_i \sum_j (\Delta \ln p_{it})(\Delta \ln p_{jt})\, [\sum_k \hat{\gamma}_{ik}\hat{\gamma}_{jk}]$.