A Homothetic Utility Function for Monopolistic Competition Models, Without Constant Price Elasticity*

by

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Abstract

The symmetric translog expenditure function leads to a demand system that has unitary income elasticity but non-constant price elasticities. This expenditure function will be useful in monopolistic competition models, and retains its properties even as the number of goods varies.

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1. Introduction

In their pioneering work on monopolistic competition, Dixit and Stiglitz (1977) proposed the symmetric, additively separable utility function $U = \sum_{i=1}^{N} \phi(c_i)$, $\phi' > 0$, $\phi'' < 0$, where c_i is the consumption of good i=1,…,N. This utility function was used by Krugman (1979), for example, in his work on the gains from international trade. However, for many problems this function is inconvenient because it is non-homothetic. So instead, most of the research that employs the Dixit-Stiglitz monopolistic competition model has used the special case $\phi(c) = c^{\theta}$, with $0<\theta<1$. This corresponds to the *constant elasticity of substitution (CES)* utility function, which is homothetic and has elasticity $\sigma = 1/(1-\theta) > 1$.

Despite its widespread use, the CES functional form has some undesirable features for monopolistic competition models. In the first place, it leads (for large N) to a constant markup of price over marginal costs. Furthermore, for several different specification of costs, this leads to a constant level of output consistent with zero profits. It follows that the equilibrium entry and exit of firms has no effect at all on the prices or output of existing firms. For example, opening a country to international trade under the CES specification does not affect the prices or scale of domestic firms, so the pro-competitive effects of international trade identified by Krugman (1979) do not apply.[1] Nevertheless, Krugman (1980,1981) uses the CES specification to obtain other results in international trade that are intractable without homotheticity, as do many other researchers in international and in macroeconomics.

Rather than choosing the functional form based on the questions being asked, it would seem desirable to have a utility function that is *both* homothetic and allows for a non-constant

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[1] Of course, even constant markups under the CES function there are still gains from trade due to increased product variety (Krugman, 1981), as well as conventional gains due to comparative advantage.

elasticity.[2] Such a function has been proposed by Bergin and Feenstra (2000, 2001). They use a *symmetric translog* expenditure function. While there is no closed-form solution for the *direct* utility function, it is homothetic, and the corresponding demand functions are easily obtained. It turns out that the price elasticity of demand is not constant, but rather, varies with the prices of competing goods: when the elasticity exceeds unity (as is necessary under monopolistic competition), then a fall in a competing price raises the elasticity, leading to a pro-competitive reduction in markups. This creates a direct linkage between the equilibrium price of each product and the prices of its competitors, in contrast to the CES case.

However, there is still a limitation of the function used by Bergin and Feenstra: they assumed that the number of goods entering the utility function was *fixed.* While this assumption is commonly used in macroeconomic models (e.g. Obstfeld and Rogoff, 1996, chap. 10), it violates the spirit of monopolistic competition under which firms enter until profits are zero. This raises the question of whether the symmetric translog indirect expenditure function *also* allows the number of goods to be varied in a convenient way. To see why this is a non-trivial issue, recall that when some goods are not purchased then their prices should be set at their *reservation* level (i.e. where demand is zero) in the expenditure function. The need to keep track of such reservations prices makes the idea of using an expenditure function in a monopolistic competition model seem too complex. So are we forced to give up this approach when the number of goods is allowed to vary?

The purpose of this note is to demonstrate a positive result: the symmetric translog function used by Bergin and Feenstra (2000, 2001) *remains valid* even when the number of goods varies. That is, once we actually solve for the reservation prices of non-consumed goods,

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[2] Various utility function used in monopolistic competition models within the industrial organization literature have non-constant elasticity, such as Spence (1976), but these are also non-homothetic.

and substitute these back into the expenditure function, we obtain a "reduced form" expenditure function that is highly tractable. In particular, the number of available goods enters the index of summation within the expenditure function, and also affects the parameters of the "reduced form" function. This is despite the fact that the parameters of the original expenditure function, defined over the universe of goods, *do not* depend on the number of available goods. The "reduced form" function that we obtain is simple enough that it should prove useful in many contexts, even when the number of goods is allowed to vary.

Our theorem on the "reduced form" expenditure function is stated in section 2, and proved in section 3. In the concluding section 4 we make further comparisons of the symmetric translog function with the CES case.

2. Translog Function

Let $\tilde{N}$ be the total number of goods conceivably available, which we treat as *fixed*. The translog expenditure function (Diewert, 1974, p. 139) is defined as:

$$\ln E = \ln U + \alpha_0 + \sum_{i=1}^{\tilde{N}} \alpha_i \ln p_i + \frac{1}{2} \sum_{i=1}^{\tilde{N}} \sum_{j=1}^{\tilde{N}} \gamma_{ij} \ln p_i \ln p_j \text{, with } \gamma_{ij} = \gamma_{ij} \,. \qquad (1)$$

Note that the restriction that $\gamma_{ij} = \gamma_{ij}$ is made without loss of generality. To ensure that the expenditure function to be homogenous of degree one, we add the restrictions that:

$$\sum_{i=1}^{\tilde{N}} \alpha_i = 1, \qquad \text{and} \qquad \sum_{i=1}^{\tilde{N}} \gamma_{ij} = 0 \,. \qquad (2)$$

In order to further require that all goods enter "symmetrically" into the expenditure function, we can impose that additional restrictions that:

$$\alpha_i = 1/\tilde{N},\ \gamma_{ii} = -\gamma(\tilde{N}-1)/\tilde{N},\ \text{and}\ \gamma_{ij} = \gamma/\tilde{N}\ \text{for } i \neq j,\ \text{with } i, j = 1,\ldots,\tilde{N}. \tag{3}$$

It is readily confirmed that the restriction in (3) satisfy the homogeneity conditions (2). Notice that while the parameters defined in (3) depend on the universal number of goods $\tilde{N}$, these parameters are fixed because $\tilde{N}$ is also.

The share of each good in expenditure can be computed by differentiating (1) with respect to $\ln p_i$, obtaining:

$$s_i = \alpha_i + \sum_{j=1}^{\tilde{N}} \gamma_{ij} \ln p_j\ . \tag{4}$$

These shares must be non-negative, of course, but we will allow for a subset of goods to have zero shares, because they are not available for purchase (i.e. have not been invented or are obsolete). To be precise, suppose that $s_i > 0$ for i=1,…,N, while $s_j = 0$ for j=N+1,…, $\tilde{N}$. Then for the latter goods, we set $s_j = 0$ within the share equations (4), and use these ($\tilde{N}$-N) equations to solve for the reservation prices $\tilde{p}_j$, j=N+1,…, $\tilde{N}$, in terms of the observed prices p_i, i=1,…,N. Then these reservation prices $\tilde{p}_j$ should appear in the expenditure function (1) for the unavailable goods j=N+1,…, $\tilde{N}$.

In the presence of unavailable goods, then, the expenditure function becomes rather complex, involving their reservation prices. However, if we consider the symmetric case defined by (3), then it turns out that the expenditure function can be simplified considerably, so that the reservation prices no longer appear explicitly. This is the point of the following result, which will be proved in the next section:

<u>Theorem</u>

Suppose that the symmetry restriction (3), with $\gamma > 0$, are imposed on the expenditure function (1). In addition, suppose that only the goods i=1,…,N are available, so that the reservation prices $\tilde{p}_j$ for j=N+1,…, $\tilde{N}$ are used in (1). Then the expenditure function can be equivalently written as:

$$\ln E = \ln U + a_0 + \sum_{i=1}^{N} a_i \ln p_i + \tfrac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij} \ln p_i \ln p_j , \tag{5}$$

where:

$$a_i = 1/N, \; b_{ii} = -\gamma(N-1)/N, \; \text{and} \; b_{ij} = \gamma/N \text{ for } i \neq j, \text{ with } i, j = 1,\ldots,N, \tag{6}$$

and,

$$a_0 = \alpha_0 + (1/2)(\tilde{N} - N)/\gamma N\tilde{N} . \tag{7}$$

Notice that the expenditure function in (5) looks like a conventional translog function defined over the goods i=1,…,N, while the symmetry restrictions in (6) ensure that expenditure is homogeneous of degree one in the observed prices p_i, i=1,…,N. The remarkable feature of this expenditure function is that it is completely valid to let N vary, as new products enter and old ones disappear. Notice that the “reduced form” parameters in (6) and (7) then vary, even though the underlying taste parameters in (3) are fixed. The last term appearing on the right of (7) reflects the welfare gain of increasing the number of available product from N to $\tilde{N}$, when the prices of all goods are set at unity.

3. Proof of Theorem

Initially we will not use the symmetry restriction (3), and write (1) in matrix form as:

$$\ln E = \ln U + \alpha_0 + \alpha' \ln p + (1/2) \ln p' \Gamma \ln p , \tag{8}$$

where α' is the row vector $(\alpha_1,...,\alpha_{\tilde{N}})$; $\ln p$ is the column vector $(\ln p_1,...,\ln p_{\tilde{N}})'$; and Γ is the symmetric matrix with elements γ_{ij}. Similarly, the share equations (4) can be written as:

$$s = \alpha + \Gamma \ln p . \tag{9}$$

Notice that substituting (9) into (8), the expenditure function can be re-written as:

$$\ln E = \ln U + \alpha_0 + (1/2)(\alpha + s)' \ln p . \tag{10}$$

Now partition the various vectors into the two sets of goods: $\alpha^{1}{}' = (\alpha_1,...,\alpha_N)$; $\alpha^{2}{}' = (\alpha_{N+1},...,\alpha_{\tilde{N}})$; $\ln p^1 = (\ln p_1,...,\ln p_N)'$; $\ln p^2 = (\ln p_{N+1},...,\ln p_{\tilde{N}})'$; $s^1 = (s_1,...,s_N)'$; $s^2 = (s_{N+1},...,s_{\tilde{N}})'$; and similarly for the matrix $\Gamma = \begin{bmatrix} \Gamma^{11} & \Gamma^{12} \\ \Gamma^{21} & \Gamma^{22} \end{bmatrix}$. As in the theorem, we assume that the goods j=N+1,…, $\tilde{N}$ are not available, so that $s^2 = 0$. It follows from (9) that the reservation prices $\ln \tilde{p}^2$ are solved as:

$$\ln \tilde{p}^2 = -(\Gamma^{22})^{-1}(\alpha^2 + \Gamma^{21} \ln p^1) . \tag{11}$$

Note that the restrictions (3) ensure that Γ^{22} is invertible, as we show by solving for its eigenvalues below. Substituting (11) into (10), and noting that $s^2 = 0$, the expenditure function becomes:

$$\begin{aligned} \ln E &= \ln U + \alpha_0 + (1/2)(\alpha^1 + s^1)' \ln p^1 + (1/2)\alpha^{2}{}' \ln \tilde{p}^2 \\ &= \ln U + \alpha_0 + (1/2)(\alpha^1 + s^1)' \ln p^1 - (1/2)\alpha^{2}{}'(\Gamma^{22})^{-1}(\alpha^2 + \Gamma^{21} \ln p^1) \\ &= \ln U + \alpha_0 + (1/2)(a^1 + s^1)' \ln p^1 - (1/2)\alpha^{2}{}'(\Gamma^{22})^{-1}\alpha^2 , \end{aligned} \tag{12}$$

where the last line is obtained by defining $a^{1}{}' \equiv \alpha^{1}{}' - \alpha^{2}{}'(\Gamma^{22})^{-1}\Gamma^{21}$. Noting that Γ is symmetric so that $\Gamma^{21}{}' = \Gamma^{12}$, we can equivalently write this definition as,

$$a^1 \equiv \alpha^1 - \Gamma^{12}(\Gamma^{22})^{-1}\alpha^2. \tag{13}$$

Also substitute (11) into the share equations for the goods i=1,..,N, to obtain:

$$\begin{aligned} s^1 &= \alpha^1 + \Gamma^{11}\ln p^1 + \Gamma^{12}\ln \tilde{p}^2 \\ &= \alpha^1 + \Gamma^{11}\ln p^1 - \Gamma^{12}(\Gamma^{22})^{-1}(\alpha^2 + \Gamma^{21}\ln p^1) \\ &= a^1 + B^{11}\ln p^1, \end{aligned} \tag{14}$$

where the final line is obtained by using (13), and defining:

$$B^{11} \equiv \Gamma^{11} - \Gamma^{12}(\Gamma^{22})^{-1}\Gamma^{21}. \tag{15}$$

Then substituting the share equation (14) into (12), we can write the expenditure function as:

$$\ln E = \ln U + \alpha_0 - (1/2)\alpha^{2\prime}(\Gamma^{22})^{-1}\alpha^2 + a^{1\prime}\ln p^1 + (1/2)\ln p^{1\prime}B^{11}\ln p^1. \tag{16}$$

By defining,

$$a_0 \equiv \alpha_0 - (1/2)\alpha^{2\prime}(\Gamma^{22})^{-1}\alpha^2, \tag{17}$$

it is evident that the expenditure function in (16) takes on a conventional translog form, defined over the prices p^1. Then to complete the proof of the theorem, we just need to show that the parameters defined in (13), (15) and (17) equal those in (6) and (7), once we make use of the original symmetry restrictions in (3).

To write these symmetry restrictions in matrix form, let L_{MxN} denote a MxN matrix with elements of unity. Then from (3), it is evident that: $\alpha^1 = (1/\tilde{N})L_{Nx1}$; $\alpha^2 = (1/\tilde{N})L_{(\tilde{N}-N)x1}$; and $\Gamma^{12} = (\gamma/\tilde{N})L_{Nx(\tilde{N}-N)}$. In addition, defining I_M as the MxM identity matrix, we have that

$\Gamma^{22} = (\gamma/\tilde{N})[-\tilde{N}\ I_{(\tilde{N}-N)} + L_{(\tilde{N}-N)x(\tilde{N}-N)}]$. Then using these expression to evaluate (13), we obtain:

$$a^1 = (1/\tilde{N})\left\{ L_{Nx1} + L_{Nx(\tilde{N}-N)}[\tilde{N}\ I_{(\tilde{N}-N)} - L_{(\tilde{N}-N)x(\tilde{N}-N)}]^{-1} L_{(\tilde{N}-N)x1} \right\}.$$

Notice that the matrix $[\tilde{N}\ I_{(\tilde{N}-N)} - L_{(\tilde{N}-N)x(\tilde{N}-N)}]$ has an eigenvector of unity, with the associated eigenvalue of N. Therefore, its inverse also has an eigenvector of unity, with the associated eigenvalue of 1/N. It follows that a^1 can be written from above as:

$$\begin{aligned} a^1 &= (1/\tilde{N})[L_{Nx1} + (1/N)L_{Nx(\tilde{N}-N)} L_{(\tilde{N}-N)x1}] \\ &= (1/\tilde{N})[L_{Nx1} + ((\tilde{N}-N)/N)L_{Nx1}] \\ &= (1/N)L_{Nx1}\ , \end{aligned}$$

where the second line follows by matrix multiplication and the third line by arithmetic. This establishes that the first restriction in (6) holds.

Now consider the definition of B^{11} in (15). Notice that from (3) we can express Γ^{11} as $\Gamma^{11} = (\gamma/\tilde{N})[-\tilde{N}\ I_N + L_{NxN}]$. Substituting this along with the earlier expression for Γ^{12} into (15), we obtain:

$$B^{11} = (\gamma/\tilde{N})[-\tilde{N}\ I_N + L_{NxN}] + (\gamma/\tilde{N})L_{Nx(\tilde{N}-N)}[\tilde{N}\ I_{(\tilde{N}-N)} - L_{(\tilde{N}-N)x(\tilde{N}-N)}]^{-1} L_{(\tilde{N}-N)xN}.$$

Again using eigenvalue properties of the inverse matrix, we evaluate this as:

$$B^{11} = (\gamma/\tilde{N})[-\tilde{N}\ I_N + L_{NxN}] + (\gamma/N\tilde{N})L_{Nx(\tilde{N}-N)}\ L_{(\tilde{N}-N)xN}$$

$$= (\gamma/\tilde{N})[-\tilde{N}\, I_N + L_{NxN}] + [\gamma(\tilde{N} - N)/N\tilde{N})]L_{NxN}$$

$$= -\gamma I_N + (\gamma/N)L_{NxN}\,,$$

where the second line again follows by matrix multiplication and the third line by arithmetic. This establishes that the second two restrictions in (6) holds.

Finally, evaluating (17) in a similar fashion, we have,

$$a_0 = \alpha_0 + (1/2)(1/\gamma\tilde{N})L_{1x(\tilde{N}-N)}[\tilde{N}\, I_{(\tilde{N}-N)} - L_{(\tilde{N}-N)x(\tilde{N}-N)}]^{-1}L_{(\tilde{N}-N)x1}$$

$$= \alpha_0 + (1/2)(1/\gamma N\tilde{N})L_{1x(\tilde{N}-N)}L_{(\tilde{N}-N)x1}$$

$$= \alpha_0 + (1/2)(\tilde{N} - N)/\gamma N\tilde{N}\,,$$

which establishes (7).

4. Conclusions

Going back to the symmetric CES case, note that the expenditure function dual to U = $\sum_{i=1}^{N} c_i^{\theta}$ is readily solved as,

$$E = U^{1/\theta} + \left(\sum_{i=1}^{N} p_i^{\theta/(\theta-1)}\right)^{(\theta-1)/\theta}. \qquad (18)$$

In principle, it would seem that the reservation prices for non-available goods should also enter this CES expenditure function. For example, suppose that good N disappears. With the elasticity of substitution greater than unity, the reservation price for that good is infinity (Feenstra, 1994). Substituting this back into the expenditure function, and noting that $0<\theta<1$, we see that the reservation price of infinitely, raised to a negative power, equals zero. So the expenditure function (18) is simply re-written with the summation running from i=1,…,N-1. Thus, even when we properly account for reservation prices, the entry and exit of goods in the CES case only affects the index of summation.

What our theorem shows is that this very simple property also applies to the symmetric, translog expenditure function: as shown in (5), the number of goods enters the index of summation, and there is no need to explicitly keep track of reservation prices. In addition, the number of goods affects the parameters in (6) and (7), despite the fact that the underlying taste parameters in (3) are fixed. So the ease of working with varying number of goods applies equally well to the symmetric CES and translog cases.

But beyond this, the translog case allows the elasticity of demand to vary with the prices and number of competing goods. To see this, note that the expenditure shares in the symmetric case can be computed by differentiating (5) and using (6), obtaining,

$$s_i = (1/N) - \gamma(N-1)\ln p_i / N + \sum_{j=1,\, j\neq i}^{N} \gamma \ln p_j / N . \tag{19}$$

Note that this can be written in the convenient form,

$$s_i = (1/N) + \gamma\left(\overline{\ln p} - \ln p_i\right), \tag{19'}$$

where $\overline{\ln p} \equiv \sum_{j=1}^{N} \ln p_j / N$. Thus, a 1% increase in the price of a product (holding the overall mean price fixed) lowers its expenditure share by γ percentage points.

The elasticity of demand ε_i is obtained as $\varepsilon_i = 1 - d\ln s_i / d\ln p_i = 1 + \gamma(N-1)/(Ns_i)$. Notice that is N is large (such as if we treat the goods available as a continuum), then this expression for the elasticity simplifies to $\varepsilon_i = 1 + (\gamma / s_i)$. Differentiating this with respect to p_i, we obtain $d\varepsilon_i / d\ln p_i = -(\gamma / s_i^2)(ds_i / d\ln p_i) = (\gamma / s_i)^2$. The restriction $\gamma > 0$ used in the theorem is needed to ensure that the elasticity exceeds unity, and it follows that the elasticity of demand is increasing in price.

The markup of prices over marginal costs can be expressed as:

$$
\begin{aligned}
(p_i / mc_i) - 1 = 1/(\varepsilon_i - 1) &= s_i N / [\gamma(N-1)] \\
&= 1/[\gamma(N-1)] - [N/(N-1)]\left(\overline{\ln p} - \ln p_i\right),
\end{aligned} \tag{20}
$$

using (19'). If markups are low, then the price-cost markup on the left of (20) approximately equals $\ln(p_i / mc_i)$. Using this approximation, then we can solve for $\ln p_i$ from (20) as,

$$
\ln p_i \approx (1/2)\ln mc_i + (1/2)\sum_{j=1, j\neq i}^{N} \ln p_j / (N-1) + 1/[2\gamma(N-1)]. \tag{21}
$$

Thus, (21) shows that the optimal prices place one-half of their weight on marginal costs, and the other half on competitor's prices. In addition, as the number of products N grows, the final term on the right of (21) falls, so prices decline due to more product variety. Both these results follow from the fact that the elasticity itself is not constant. This is in stark contrast to the CES case, where prices (for large N) are a *fixed* markup over marginal costs, and do not depend on competitor's prices at all. Of course, in some models there is no need to work with the approximation leading to (21), and the exact form of the pricing equation in (20) can be used instead. In either case, the strong linkage between the optimal price of a good and the prices of its competitors will apply.

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