Appendix to Export Variety and Country Productivity:
ESTIMATING THE MONOPOLISTIC COMPETITION MODEL WITH ENDOGENOUS PRODUCTIVITY
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For reference, we begin by repeating certain equations from the main text, but using a numbering scheme unique to this Appendix. Letting M_{ie} denote the mass of new entrants in sector i, then $[1-G_i(\varphi_i^*)]M_{ie}$ firms successfully produce. In a stationary equilibrium, these should replace the firms going bankrupt, so that:

$$[1-G_i(\varphi_i^*)]M_{ie} = \delta M_i \quad . \tag{1}$$

Conditional on successful entry, the distribution of productivities for firms in sector i is then:

$$\mu_i(\varphi_i) = \begin{cases} \dfrac{g_i(\varphi_i)}{[1-G_i(\varphi_i^*)]} & \text{if } \varphi_i \geq \varphi_i^*, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where $g_i(\varphi_i) = \partial G_i(\varphi_i)/\partial \varphi_i$.

Revenue earned by a home firm selling in the domestic sector i at the price $p_i(\varphi_i)$ is:

$$r_i(\varphi_i) = p_i(\varphi_i)q_i(\varphi_i) = \left[\frac{p_i(\varphi_i)}{P_i^H}\right]^{1-\sigma_i} E_i^H, \quad \sigma_i > 1, \tag{3}$$

where $q_i(\varphi_i)$ is the quantity sold and P_i^H is the home CES price index for sector i:

$$P_i^H = \left[\int_{\varphi_i^*}^{\infty} p_i(\varphi_i)^{1-\sigma_i} M_i \mu_i(\varphi_i) d\varphi_i + \int_{\varphi_{ix}^{F*}}^{\infty} p_i^F(\varphi_i)^{1-\sigma_i} M_i^F \mu_i^F(\varphi_i) d\varphi_i \right]^{\frac{1}{1-\sigma_i}} . \quad (4)$$

Using the second equality in (3) to solve for price as a function of quantity, and substituting this back into (3), we obtain:

$$r_i(\varphi_i) = A_{id}\, q_i(\varphi_i)^{\frac{\sigma_i - 1}{\sigma_i}}, \text{ where } A_{id} \equiv P_i^H \left(\frac{E_i^H}{P_i^H} \right)^{\frac{1}{\sigma_i}} . \quad (3')$$

Letting $p_{ix}(\varphi_i)$ and $q_{ix}(\varphi_i)$ denote the price received and quantity shipped at the *factory-gate*, the revenue earned by the exporter is:

$$r_{ix}(\varphi_i) = p_{ix}(\varphi_i) q_{ix}(\varphi_i) = \left[\frac{p_{ix}(\varphi_i)\tau_i}{P_i^F} \right]^{1-\sigma_i} E_i^F , \quad (5)$$

where P_i^F is the aggregate CES price for sector i in the foreign country, and E_i^F is foreign expenditure in sector i. Once again, it is convenient to treat revenue as a function of quantity, which is determined from (5) as:

$$r_{ix}(\varphi_i) = A_{ix} q_{ix}(\varphi_i)^{\frac{\sigma_i - 1}{\sigma_i}}, \text{ where } A_{ix} \equiv \left(\frac{P_i^F}{\tau_i} \right) \left(\frac{\tau_i E_i^F}{P_i^F} \right)^{\frac{1}{\sigma_i}} . \quad (5')$$

Integrating (3') and (5'), we obtain revenue from domestic and export sales in sector i:

$$R_{id} \equiv M_i A_{id} \int_{\varphi_i^*}^{\infty} q_i(\varphi_i)^{\frac{\sigma_i - 1}{\sigma_i}} \mu_i(\varphi_i) d\varphi_i , \quad (6)$$

$$R_{ix} \equiv M_i A_{ix} \int_{\varphi_{ix}^*}^{\infty} q_{ix}(\varphi_i)^{\frac{\sigma_i - 1}{\sigma_i}} \mu_i(\varphi_i) d\varphi_i . \quad (7)$$

Notice that the mass of domestic firms or varieties is $\int_{\varphi_i^*}^{\infty} M_i \mu_i(\varphi_i) d\varphi_i = M_i$, which we are aggregating over in (6). But the mass of exporting firms or varieties that we are aggregating over in (7) is $\int_{\varphi_{ix}^*}^{\infty} M_i\mu_i(\varphi_i)d\varphi_i \le M_i$. It will be convenient to denote the range of *export varieties relative to domestic varieties* by:

$$\chi_i \equiv \left(\int_{\varphi_{ix}^*}^{\infty} \mu_i(\varphi)d\varphi \right) = \left(\frac{1 - G(\varphi_{ix}^*)}{1 - G(\varphi_i^*)} \right) \le 1 . \tag{8}$$

Not explained in the main text are the *resource constraints* for the economy, which are as follows. Having M_i firms produce output $q_i(\varphi_i)$ for the domestic market has a cost of $h_i[v_i(\varphi_i)] = M_i[(q_i(\varphi_i)/\varphi_i) + f_i]$, where f_i is the fixed cost of production, $v_i(\varphi_i)$ is a K-dimensional vector of factor demand, and $h_i : R^K \to R^1$ is a homogeneous of degree one and strictly quasi-concave mapping from the vector of factor demands to a scalar. The total resources used for domestic production are then:

$$\int_{\varphi_i^*}^{\infty} h_i[v_i(\varphi_i)]\mu_i(\varphi_i)d\varphi_i = \int_{\varphi_i^*}^{\infty} M_i[(q_i(\varphi_i)/\varphi_i) + f_i]\mu_i(\varphi_i)d\varphi_i . \tag{9}$$

Likewise, the exporting firms producing output q_{ix} with productivity φ_i have a resource cost of:

$$\int_{\varphi_{ix}^*}^{\infty} h_i[v_{ix}(\varphi_i)]\mu_i(\varphi_i)d\varphi_i = \int_{\varphi_{ix}^*}^{\infty} M_i[(q_{ix}(\varphi_i)/\varphi_i) + f_{ix}]\mu_i(\varphi_i)d\varphi_i , \tag{10}$$

where f_{ix} is the additional fixed cost for exporters. We assume that the function h_i for export and domestic production is the same, so that the factor proportions in the two activities are equal.

Total GDP in the economy is obtained by summing revenue over the sectors:

$$R = \sum_{i=1}^{N} R_{id} + R_{ix} . \tag{11}$$

Consider now a social planner's problem of maximizing GDP in (11), subject to the resource constraints for the economy, which are (9), (10) and:

$$\sum_{i=1}^{N}\left[\int_{\varphi_i^*}^{\infty} v_i(\varphi_i)\mu_i(\varphi_i)d\varphi_i + \int_{\varphi_{ix}^*}^{\infty} v_{ix}(\varphi_i)\mu_i(\varphi_i)d\varphi_i\right] = V - \sum_{i=1}^{N} M_{ie}v_{ie}. \tag{12}$$

The summation on the left of (12) is total resources used in production. On the right of (12), V is the vector of factor endowments for the economy, and from that we subtract the vector of fixed costs v_{ie} from entry into each sector, times the mass of entering firms M_{ie}. The number of entering firms is given by (1), which we can substitute into (12), obtaining:

$$\sum_{i=1}^{N}\left[\int_{\varphi_i^*}^{\infty} v_i(\varphi_i)\mu_i(\varphi_i)d\varphi_i + \int_{\varphi_{ix}^*}^{\infty} v_{ix}(\varphi_i)\mu_i(\varphi_i)d\varphi_i\right] = V - \sum_{i=1}^{N}\frac{\delta M_i}{[1-G_i(\varphi_i^*)]}v_{ie} \quad . \tag{13}$$

In maximizing GDP we will hold fixed the shift parameters A_{id} and A_{ix} , which is analogous to holding product prices constant in a conventional GDP function. Then we have:

Proposition 1

Choose $q_i(\varphi_i)$, $q_{ix}(\varphi_i)$, $v_i(\varphi_i)$, $v_{ix}(\varphi_i)$, φ_i^*, φ_{ix}^*, and M_i for i = 1,…,N, to maximize GDP in (11), subject to (9), (10) and (13). Holding fixed the shift parameters $A_d = (A_{1d},\ldots,A_{Nd})$ and $A_x = (A_{1x},\ldots,A_{Nx})$, the first-order conditions for an interior maximum are identical to the equilibrium conditions for the monopolistically competitive economy. Then GDP can be written as a function $R(A_d, A_x, V)$, and satisfies:

(a) $R(A_d, A_x, V)$ is homogeneous of degree one in (A_d, A_x), and if R is differentiable then,

$$\frac{\partial \ln R}{\partial \ln A_{id}} = \frac{R_{id}}{R}, \text{ and } \frac{\partial \ln R}{\partial \ln A_{ix}} = \frac{R_{ix}}{R}; \tag{14}$$

(b) $R(A_d, A_x, V)$ is homogeneous of degree one in V, and $\partial R / \partial V = w$ which is the vector of factor prices.

Proof:

Introduce m_i as the Lagrange multiplier on constraint (9), m_{ix} as the Lagrange multiplier on (10), and the vector w as the multipliers on (13). Then the Lagrangian can be written as:

$$L = \sum_{i=1}^{N} \left\{ \int_{\varphi_i^*}^{\infty} M_i A_{id} q_i(\varphi_i)^{\frac{\sigma_i - 1}{\sigma_i}} \mu_i(\varphi_i) d\varphi_i + \int_{\varphi_{ix}^*}^{\infty} M_i A_{ix} q_{ix}(\varphi_i)^{\frac{\sigma_i - 1}{\sigma_i}} \mu_i(\varphi_i) d\varphi_i \right\}$$

$$+ \sum_{i=1}^{N} m_i \left\{ \int_{\varphi_i^*}^{\infty} h_i[v_i(\varphi_i)] \mu(\varphi_i) d\varphi_i - \int_{\varphi_i^*}^{\infty} M_i[(q_i / \varphi_i) + f_i] \mu(\varphi_i) d\varphi_i \right\}$$

$$+ \sum_{i=1}^{N} m_{ix} \left\{ \int_{\varphi_{ix}^*}^{\infty} h_i[v_{ix}(\varphi_i)] \mu(\varphi_i) d\varphi_i - \int_{\varphi_{ix}^*}^{\infty} M_i[(q_{ix} / \varphi_i) + f_{ix}] \mu(\varphi_i) d\varphi_i \right\}$$

$$+ w' \left\{ V - \sum_{i=1}^{N} \frac{\delta M_i}{[1 - G_i(\varphi_i^*)]} v_{ie} - \sum_{i=1}^{N} \left[\int_{\varphi_i^*}^{\infty} v_i(\varphi_i) \mu_i(\varphi_i) d\varphi_i + \int_{\varphi_{ix}^*}^{\infty} v_{ix}(\varphi_i) \mu_i(\varphi_i) d\varphi_i \right] \right\}$$

$$= \sum_{i=1}^{N} \int_{\varphi_i^*}^{\infty} M_i \left\{ A_{id} q_i(\varphi_i)^{\frac{\sigma_i - 1}{\sigma_i}} - m_i \left[\frac{q_i(\varphi_i)}{\varphi_i} + f_i \right] \right\} \mu_i(\varphi_i) d\varphi_i$$

$$+ \sum_{i=1}^{N} \int_{\varphi_{ix}^*}^{\infty} M_i \left\{ A_{ix} q_{ix}(\varphi_i)^{\frac{\sigma_i - 1}{\sigma_i}} - m_{ix} \left[\frac{q_{ix}(\varphi_i)}{\varphi_i} + f_{ix} \right] \right\} \mu_i(\varphi_i) d\varphi_i$$

$$+ \sum_{i=1}^{N} \left\{ \int_{\varphi_i^*}^{\infty} [m_i h_i(v_i(\varphi_i)) - w' v_i(\varphi_i)] \mu_i(\varphi_i) d\varphi_i \right\} \tag{A1}$$

$$+ \sum_{i=1}^{N} \left\{ \int_{\varphi_{ix}^*}^{\infty} [m_{ix} h_{ix}(v_{ix}(\varphi_i)) - w' v_{ix}(\varphi_i)] \mu_i(\varphi_i) d\varphi_i \right\} + w' \left\{ V - \sum_{i=1}^{N} \frac{\delta M_i v_{ie}}{[1 - G_i(\varphi_i^*)]} \right\}$$

where (A1) follows by grouping terms.

By inspection of the Lagrangian, the maximized value of GDP is a function $R(A_d, A_x, V)$. The first-order condition with respect to the domestic quantity $q_i(\varphi_i)$ is obtained by differentiating the terms within the first integral in (A1), and yields:

$$\left(\frac{\sigma_i - 1}{\sigma_i}\right) A_{id} q_i(\varphi_i)^{-\frac{1}{\sigma_i}} = \frac{m_i}{\varphi_i}. \tag{A2}$$

The expression $A_{id} q_i(\varphi_i)^{-\frac{1}{\sigma_i}}$ equals price $p_i(\varphi_i)$, as can be seen from the demand function (3), so this first-order condition is just marginal revenue equals marginal cost:

$$\left(\frac{\sigma_i - 1}{\sigma_i}\right) p_i(\varphi_i) = \frac{m_i}{\varphi_i}. \tag{A3}$$

Likewise, maximization of the Lagrangian with respect to the export quantity $q_{ix}(\varphi_i)$, in the second integral of (A1), also yields marginal revenue equal to marginal cost, for exports:

$$\left(\frac{\sigma_i - 1}{\sigma_i}\right) p_{ix}(\varphi_i) = \frac{m_{ix}}{\varphi_i}. \tag{A4}$$

Using (A3) and (A4), we can calculate that the profits from domestic and export sales are,

$$r_i(\varphi_i) - \left(\frac{m_i}{\varphi_i}\right) q_i(\varphi_i) = \frac{r_i(\varphi_i)}{\sigma_i}, \text{ and } r_{ix}(\varphi_i) - \left(\frac{m_{ix}}{\varphi_i}\right) q_{ix}(\varphi_i) = \frac{r_{ix}(\varphi_i)}{\sigma_i}. \tag{A5}$$

Maximization with respect to $v_i(\varphi_i)$ and $v_{ix}(\varphi_i)$ in the third and forth integrals yields,

$$m_i \frac{\partial h_i(v_i(\varphi_i))}{\partial v_i(\varphi_i)} = w, \tag{A6}$$

and,

$$m_{ix} \frac{\partial h_i(v_{ix}(\varphi_i))}{\partial v_{ix}(\varphi_i)} = w. \tag{A7}$$

Therefore:

$$\frac{\partial h_i(v_i(\varphi_i))}{\partial v_{ik}(\varphi_i)} \Bigg/ \frac{\partial h_i(v_i(\varphi_i))}{\partial v_{i\ell}(\varphi_i)} = w_k / w_\ell = \frac{\partial h_i(v_{ix}(\varphi_i))}{\partial v_{ixk}(\varphi_i)} \Bigg/ \frac{\partial h_i(v_{ix}(\varphi_i))}{\partial v_{ix\ell}(\varphi_i)}. \tag{A8}$$

Since the function h_i is assumed to be strictly quasi-concave, it follows from (A8) that the ratio of demand for factors k and ℓ are identical in domestic and export use. Therefore, the values of v_i and v_{ix} are multiples of each other, $v_i = \lambda_i v_{ix}$. But since h_i homogeneous of degree one, its first derivative is homogeneous of degree zero, so any solution $v_i = \lambda_i v_{ix}$ in (A6) yields exactly the same value for the derivatives $\partial h_i(v_i)/\partial v_i$ as does $\partial h_i(v_{ix})/\partial v_{ix}$ in (A7). It follows that the equalities in (A6) and (A7) can hold if and only if $m_i = m_{ix}$, so the marginal costs of domestic production and exporting are equal. Furthermore, multiplying (A6) and (A7) by v_i and v_{ix}, we immediately obtain $[m_i h_i(v_i) - w' v_i] = [m_i h_i(v_{ix}) - w' v_{ix}] = 0$.

Substituting these relations into (A1), and using (A5), we can rewrite the Lagrangian as:

$$\sum_{i=1}^{N}\left[\int_{\varphi_i^*}^{\infty} M_i\left(\frac{r_i(\varphi_i)}{\sigma_i} - m_i f_i\right)\mu_i(\varphi_i)d\varphi + \int_{\varphi_{ix}^*}^{\infty} M_i\left(\frac{r_{ix}(\varphi_i)}{\sigma_i} - m_i f_{ix}\right)\mu_i(\varphi_i)d\varphi\right] + w'\left[V - \sum_{i=1}^{N}\frac{\delta M_i v_{ie}}{[1 - G_i(\varphi_i^*)]}\right]$$

Differentiating this Lagrangian with respect to the export cutoff productivity φ_{ix}^*, we obtain $r_{ix}(\varphi_{ix}^*)/\sigma_i = m_i f_{ix}$, which states that the profits earned by the marginal exporter should just cover fixed costs. This is an equilibrium condition in Melitz (2003). Differentiating the Lagrangian with respect to M_i, we obtain:

$$\frac{[1 - G_i(\varphi_i^*)]}{\delta}\left[\int_{\varphi_i^*}^{\infty}\left(\frac{r_i(\varphi_i)}{\sigma_i} - m_i f_i\right)\mu_i(\varphi_i)d\varphi + \int_{\varphi_{ix}^*}^{\infty}\left(\frac{r_{ix}(\varphi_i)}{\sigma_i} - m_i f_{ix}\right)\mu_i(\varphi_i)d\varphi\right] = w' v_{ie}, \quad \text{(A9)}$$

where the term in brackets is the average profits earned by a successful entrant. This condition states that expected discounted profits equal the fixed costs of entry, which is the free-entry condition in Melitz (2003).

Finally, differentiating with respect to the domestic cutoff productivity φ_i^*, we need to recognize that the marginal distributions $\mu_i(\varphi_i)$ are divided by $[1-G_i(\varphi_i^*)]$. Taking into account the derivative of this term with respect to φ_i^*, and then using the free-entry condition (A9), we obtain $r_i(\varphi_i^*)/\sigma_i = m_i f_i$, so the profits earned by the marginal domestic producer with productivity φ_i^* just cover fixed costs.

Part (a) of Proposition 1 follows by differentiating (A1) with respect to A_{id}, obtaining:

$$\frac{\partial L}{\partial A_{id}} = \frac{\partial R}{\partial A_{id}} = \int_{\varphi_i^*}^{\infty} M_i q_i(\varphi_i)^{\frac{\sigma_i - 1}{\sigma_i}} \mu_i(\varphi_i) d\varphi_i = R_i / A_{id} .$$

It follows that $\partial \ln R / \partial \ln A_{id} = (R_{id} / R)$, and a similar condition holds for exports. Multiplying all A_{id} and A_{ix} in (A1)–(A4) by $\lambda > 0$ will increase prices p_i and p_{ix}, wages w and marginal costs $m_i = m_{ix}$ by that amount, with no change in any quantities. It follows that the objective function is increased by $\lambda > 0$, so $R(A_d, A_x, V)$ is homogeneous of degree one in (A_d, A_x).

The result that $\partial R / \partial V = w$ in part (b) is a property of Lagrange multipliers. Multiplying the endowments V in (A1)–(A7) by $\lambda > 0$ will increase the domestic variety M_i in each sector as well as v_i and v_{ix} by that amount, with no other change in prices or quantities. It follows that the objective function is increased by $\lambda > 0$, so $R(A_d, A_x, V)$ is homogeneous of degree one in V.

QED

Following Helpman, Melitz and Yeaple (2004) and Chaney (2005), we assume the Pareto distribution for productivities, defined by:

$$G_i(\varphi_i) = 1 - \varphi_i^{-\theta_i}, \text{ with } \theta_i > \sigma_i - 1. \qquad (15)$$

The parameter θ_i is a measure of dispersion of the Pareto distribution, with lower θ_i having more weight in the upper tail. Using this distribution, we calculate export relative to domestic variety in (8) as:

$$\chi_i = \left(\frac{1-G(\varphi_{ix}^*)}{1-G(\varphi_i^*)}\right) = \left(\frac{\varphi_i^*}{\varphi_{ix}^*}\right)^{\theta_i}. \tag{16}$$

Relative export variety can be further simplified by using (3) and (5) to compute $r_i(\varphi_i^*)$ and $r_{ix}(\varphi_{ix}^*)$, together with the equilibrium conditions (A3), (A4) and the zero-cutoff profits conditions discussed in the proof of Proposition 1:

$$r_i(\varphi_i^*)/\sigma_i = m_i f_i, \quad \text{and} \quad r_{ix}(\varphi_{ix}^*)/\sigma_i = m_i f_{ix}. \tag{17}$$

Then we obtain:

$$\frac{r_{ix}(\varphi_{ix}^*)}{r_i(\varphi_i^*)} = \left(\frac{\varphi_{ix}^* P_i^F/\tau_i}{\varphi_i^* P_i^H}\right)^{\sigma_i - 1} \frac{E_i^F}{E_i^H} = \frac{f_{ix}}{f_i}.$$

Raising this expression to the power ($1/\sigma_i$), and re-arranging, we have:

$$\left(\frac{A_{ix}}{A_{id}}\right) = \chi_i^{\frac{\sigma_i - 1}{\sigma_i \theta_i}} \left(\frac{f_{ix}}{f_i}\right)^{1/\sigma_i}, \qquad 0 < \frac{\sigma_i - 1}{\sigma_i \theta_i} < 1. \tag{18}$$

Thus, the ratio of the export/domestic shift parameters in demand equals relative export variety raised to a positive power, adjusted for a term involving fixed costs. This means that relative export variety can be used as a proxy for (A_{ix}/A_{id}).

The Pareto distribution further enables us to aggregate domestic and export sales in each sector. In particular, we shown in the proof of Proposition 2 (below) that the Pareto distribution, along with (17), allows us to write export sales relative to domestic sales in each sector as:

$$\frac{R_{ix}}{R_{id}} = \chi_i \left(\frac{f_{ix}}{f_i}\right). \tag{19}$$

As one might expect, relative export variety is directly related to relative export sales. Then substituting from (18), the ratio of export to domestic sales becomes:

$$\frac{R_{ix}}{R_{id}} = \left(\frac{A_{ix}}{A_{id}}\right)^{\frac{\sigma_i\theta_i}{(\sigma_i-1)}} \left(\frac{f_{ix}}{f_i}\right)^{1-\frac{\theta_i}{(\sigma_i-1)}}. \tag{20}$$

Thus, the sales ratio is a constant-elasticity function of the relative export shift parameters. This implies that: first, the shift parameters (A_{ix}, A_{id}) for sector i are weakly separable from all other variables in the GDP function; and second, the appropriate aggregator for (A_{ix}, A_{id}) is a CES function. These results are summarized by:

Proposition 2

Assume that the distribution of firm productivity in Pareto, as in (15). Then the domestic and export parameters (A_{id}, A_{ix}) can be aggregated into a CES function:

$$\psi_i(A_{id}, A_{ix}) \equiv \left[A_{id}^{\frac{\sigma_i\theta_i}{(\sigma_i-1)}} + A_{ix}^{\frac{\sigma_i\theta_i}{(\sigma_i-1)}} (f_{ix}/f_i)^{1-\frac{\theta_i}{(\sigma_i-1)}} \right]^{\frac{(\sigma_i-1)}{\sigma_i\theta_i}}. \tag{21}$$

It follows that GDP can be written as $R(\psi_1,...,\psi_N, V)$, and if R is differentiable then:

$$\frac{\partial \ln R}{\partial \ln \psi_i} = \frac{(R_{id} + R_{ix})}{R}, \tag{22}$$

which is the share of sector i in GDP.

Proof:

From (A3) and (A4), prices are inversely proportional to productivities φ_i, so from (3) and (5) the revenue earned by firms of various productivities satisfies $r(\varphi_i'')/r(\varphi_i') = (\varphi_i''/\varphi_i')^{\sigma_i-1}$. For example, compared to the cut-off productivity φ_i^*, we have $r_i(\varphi_i) = (\varphi_i/\varphi_i^*)^{\sigma_i-1} r_i(\varphi_i^*)$. Using this relation and (3') to evaluate $r_i(\varphi_i^*)$, total revenue earned from domestic sales in sector i is:

$$R_{id} = \int_{\varphi_i^*}^{\infty} M_i r_i(\varphi_i)\mu_i(\varphi)d\varphi = M_i A_{id}\left[\frac{\tilde{\varphi}_i(\varphi_i^*)}{\varphi_i^*}\right]^{\sigma_i-1} q_i(\varphi_i^*)^{\frac{\sigma_i-1}{\sigma_i}}, \tag{A10}$$

where $\tilde{\varphi}_i(\varphi_i^*)$ is the average productivity across firms, defined as in Melitz (2003) by:

$$\tilde{\varphi}_i(\varphi_i^*) \equiv \left(\int_{\varphi_i^*}^{\infty} \varphi_i^{\sigma_i-1}\mu_i(\varphi)d\varphi\right)^{\frac{1}{(\sigma_i-1)}}. \tag{A11}$$

Equation (A10) shows that the domestic sales of home firms is equal to the sales from a mass M_i of representative firms, all with productivity $\tilde{\varphi}_i(\varphi_i^*)/\varphi_i^*$.

On the export side, it follows using the same steps as above that revenue equals $r_{ix}(\varphi_{ix}) = (\varphi_{ix}/\varphi_{ix}^*)^{\sigma_i-1} r_{ix}(\varphi_{ix}^*)$. Then using (5') to evaluate $r_{ix}(\varphi_{ix}^*)$, total revenue earned from export sales in sector i is:

$$R_{ix} = \int_{\varphi_{ix}^*}^{\infty} M_i r_{ix}(\varphi_i)\mu_i(\varphi)d\varphi = \chi_i\, M_i A_{ix}\left[\frac{\tilde{\varphi}_{ix}(\varphi_{ix}^*)}{\varphi_{ix}^*}\right]^{\sigma_i-1} q_{ix}(\varphi_{ix}^*)^{\frac{\sigma_i-1}{\sigma_i}}, \tag{A12}$$

where $\tilde{\varphi}_{ix}(\varphi_{ix}^*)$ is the average productivity across exporting firms, defined analogously to (A11) but with the cutoff productivity φ_{ix}^*:

$$\tilde{\varphi}_{ix}(\varphi_{ix}^*) \equiv \left(\int_{\varphi_{ix}^*}^{\infty} \varphi_i^{\sigma_i-1}\frac{g_i(\varphi_i)}{[1-G_i(\varphi_{ix}^*)]}d\varphi_i\right)^{\frac{1}{(\sigma_i-1)}}. \tag{A13}$$

We can think of the terms $[\tilde{\varphi}_i(\varphi_i^*)/\varphi_i^*]$ or $[\tilde{\varphi}_{ix}(\varphi_{ix}^*)/\varphi_{ix}^*]$ as a measure of the skewness of productivity. Calculating the average productivities using the Pareto distribution, we obtain:

$$\left[\frac{\tilde{\varphi}_i(\varphi_i^*)}{\varphi_i^*}\right]^{\sigma_i-1} = \left[\frac{\tilde{\varphi}_{ix}(\varphi_{ix}^*)}{\varphi_{ix}^*}\right]^{\sigma_i-1} = \left(\frac{\theta_i}{\theta_i-\sigma_i+1}\right), \quad \theta_i > \sigma_i - 1. \tag{A14}$$

Thus, the skewness of the Pareto distribution is independent of the cutoff productivity.

Substituting (A14) into (A10) and (A12), dividing these and using (17), we obtain (19) above, from which (20) is obtained. These show that the export relative to domestic share in each sector is independent of the shift parameters in all other sectors and of factor endowments. Therefore, the parameters (A_{id}, A_{ix}) in the GDP function are weakly separable from all other shift parameters and from the endowments. It follows that GDP can be written as a function $R[\psi_1(A_{1d}, A_{1x}), \ldots, \psi_N(A_{Nd}, A_{Nx}), V]$, for some linearly homogeneous functions ψ_i, i=1,…,N.

Furthermore, (20) proves that the ψ_i are CES functions, $\psi_i(A_{id}, A_{ix}) = \left(A_{id}^{\alpha} + \beta A_{ix}^{\alpha}\right)^{1/\alpha}$, for some parameters α and β. Then using Proposition 1(a) to calculate (20), we obtain:

$$\frac{R_{ix}}{R_{id}} = \frac{\left(\dfrac{\partial \ln R}{\partial \ln \psi_i}\right)\left(\dfrac{\partial \ln \psi_i}{\partial \ln A_{ix}}\right)}{\left(\dfrac{\partial \ln R}{\partial \ln \psi_i}\right)\left(\dfrac{\partial \ln \psi_i}{\partial \ln A_{id}}\right)} = \beta\left(\frac{A_{ix}}{A_{id}}\right)^{\alpha} = \left(\frac{A_{ix}}{A_{id}}\right)^{\frac{\sigma_i\theta_i}{(\sigma_i-1)}}\left(\frac{f_{ix}}{f_i}\right)^{1-\frac{\theta_i}{(\sigma_i-1)}}. \tag{A15}$$

Therefore, $\alpha = \sigma_i\theta_i/(\sigma_i-1)$ and $\beta = (f_{ix}/f_i)^{1-\frac{\theta_i}{(\sigma_i-1)}}$, from which (21) is obtained.

To obtain (22), we use (14) and $R[\psi_1(A_{1d}, A_{1x}), \ldots, \psi_N(A_{Nd}, A_{Nx}), V]$ to compute:

$$\frac{\partial \ln R}{\partial \ln A_{id}} + \frac{\partial \ln R}{\partial \ln A_{ix}} = \frac{R_{id}+R_{ix}}{R} = \left(\frac{\partial \ln R}{\partial \ln \psi_i}\right)\left[\left(\frac{\partial \ln \psi_i}{\partial \ln A_{id}}\right) + \left(\frac{\partial \ln \psi_i}{\partial \ln A_{ix}}\right)\right]. \tag{A16}$$

The final term in brackets equals unity for the CES function, so that (22) follows. **QED**