Journal of Economic Behavior and Organization, 51, 2003, 459-485.

Appendix
A Market-Power Model of Business Groups

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Revised, November 2001

To solve for the equilibria, we will make use of the full-employment condition, which in written in the symmetric equilibrium as:

$$\begin{aligned} L = & GN_b k_{yb} + GN_b \beta y_b \phi_b + GM_b k_{xb} + GM_b x_b \\ & + GM_b \tilde{x}_b + G\alpha + N_c k_{yc} + N_c \beta y_c \phi_c + M_c k_{xc} + M_c x_c . \end{aligned} \tag{A1}$$

Using (5), (9), (14) and (18), this is simplified as,

$$\begin{aligned} L = & G\left[N_b k_{yb}\left(1+\beta(\eta-1)\right) + M_b k_{xb}\sigma + \alpha\right] \\ & + N_c k_{yc}\left(1+\beta(\eta-1)\right) + M_c k_{xc}\sigma. \end{aligned} \tag{A2}$$

Another useful relation is the equality of GNP as total factor income and the value of final sales,

$$L = GN_b q_b y_b + N_c q_c y_c . \tag{A3}$$

Using (8), (9), (12) and (14) this is written in the symmetric equilibrium as,

$$L = GN_b k_{yb}\left[\eta + \left(\frac{s_{yb}}{1-s_{yb}}\right)\right] + N_c k_{yc}\eta . \tag{A4}$$

We should also indicate the values for internal and external sales of inputs, x_{bi} and $\tilde{x}_{bi}$. *External* sales of each input variety shown in (6) can be written as $\tilde{x}_{bi} = \tilde{x}_{bbi} + \tilde{x}_{bci}$, where $\tilde{x}_{bci}$ is sales to other groups and $\tilde{x}_{bci}$ is sales to downstream unaffiliated firms. Computing the derivative of unit-costs, the sales to other business groups are,

$$\tilde{x}_{bbi} = \sum_{j=1, j\neq i}^{G} p_{bi}^{-\sigma} \left[\frac{y_{bj}\phi_{bj}(1-\beta)N_{bj}}{M_{bj} + \sum_{i=1, i\neq j}^{G} M_{bi} p_{bi}^{1-\sigma} + \sum_{i=1}^{M_c} p_{ci}^{1-\sigma}} \right]$$

$$= p_b^{-\sigma} \left[\frac{(G-1) y_b \phi_b (1-\beta) N_b}{M_b + (G-1) M_b p_b^{1-\sigma} + M_c p_c^{1-\sigma}} \right] \tag{A5a}$$

where the second line applies in the symmetric equilibrium. The sales to unaffiliated firms are,

$$\tilde{x}_{bci} = \sum_{j=1}^{N_c} p_{bi}^{-\sigma} \left[\frac{y_{cj}\phi_c(1-\beta)}{\sum_{i=1}^{G} M_{bi} p_{bi}^{1-\sigma} + \sum_{i=1}^{M_c} p_{ci}^{1-\sigma}} \right]$$

$$= p_b^{-\sigma} \left[\frac{y_c \phi_c (1-\beta) N_c}{G M_b p_b^{1-\sigma} + M_c p_c^{1-\sigma}} \right] \tag{A5b}$$

where the second line applies in the symmetric equilibrium. Similarly, *internal* sales of each product variety are,

$$x_{bi} = \frac{y_{bi}\phi_{bi}(1-\beta)N_{bi}}{\left[M_{bi} + \sum_{j=1, j\neq i}^{G} M_{bj} p_{bj}^{1-\sigma} + \sum_{j=1}^{M_c} p_{cj}^{1-\sigma}\right]}$$

$$= \frac{y_b \phi_b (1-\beta) N_b}{\left[M_b + (G-1) M_b p_b^{1-\sigma} + M_c p_c^{1-\sigma}\right]} \tag{A6}$$

where the second line applies in the symmetric equilibrium.

Derivation of equation (18) in the text:

When the business group choose product variety optimally, then,

$$x_b + \tilde{x}_b = (\sigma - 1)k_{xb}. \tag{18}$$

Proof:

We need to differentiate (1) with respect to M_{bi} and set this equal to zero. Since profits in (1) have been maximized with respect to the markup μ_{bi}, it is legitimate to consider a small change $d\mu_{bi}$ chosen to ensure that the change in M_{bi} has *no effect of the final-goods price* $q_{bi} = \mu_{bi}\phi_{bi}$ charged by group i. Specifically, we will choose $d\mu_{bi}$ such that:

$$dq_{bi} = \mu_{bi} d\phi_{bi} + d\mu_{bi}\phi_{bi} = 0 \quad \Rightarrow \quad d\mu_{bi} / d\phi_{bi} = -(\mu_{bi} / \phi_{bi}).$$

Suppose first that there are no external sales of the intermediate inputs by group i, $\tilde{x}_{bi} = 0$. Since the price q_{bi} is constant, it follows that the demand y_{bi} for each final good of group i is also constant, so that change in M_{bi} only affects costs. Then from (1) we have,

$$\begin{aligned} \frac{d\Pi_{bi}}{dM_{bi}} &= N_{bi}[y_{bi}(\mu_{bi} - 1) + \frac{d\mu_{bi}}{d\phi_{bi}} y_{bi}\phi_{bi}] \frac{\partial \phi_{bi}}{\partial M_{bi}} - k_{xb} \\ &= -y_{bi} N_{bi} \frac{\partial \phi_{bi}}{\partial M_{bi}} - k_{xb} = \frac{x_{bi}}{(\sigma - 1)} - k_{xb}, \end{aligned} \tag{A7}$$

where the second equality follows by using $d\mu_{bi} / d\phi_{bi} = -(\mu_{bi} / \phi_{bi})$, and the final equality follows from differentiating (2) and comparing the result to (A6). Then (18) follows by setting (A7) equal to zero, and using symmetry.

Now suppose that there are sales of the inputs outside the group. From (2), the costs to other groups depends on $M_{bi}p_{bi}^{1-\sigma}$, chosen by group i. Since profits in (1) are maximized with respect to p_{bi}, we can consider a small change in p_{bi} designed to keep this magnitude constant,

$$dM_{bi}p_{bi}^{1-\sigma} + (1-\sigma)M_{bi}p_{bi}^{-\sigma}dp_{bi} = 0\,. \tag{A8}$$

We will consider the optimal choice of M_{bi} with p_{bi} adjusting as in (A8), which ensures that the costs of the other groups and therefore their own final-goods prices are constant. With q_{bi} and q_{bj} all constant, then demand y_{bi} is also constant. The change in profits of group i is,

$$\left.\frac{d\Pi_{bi}}{dM_{bi}}\right|_{(A8)} = \frac{\partial \Pi_{bi}}{\partial M_{bi}} + \left.\frac{d\Pi_{bi}}{dp_{bi}}\right|_{(A8)} \left.\frac{dp_{bi}}{dM_{bi}}\right|_{(A8)}.$$

Using (1) and the same steps as in (A7) and (A8), this is evaluated as,

$$\begin{aligned}
\left.\frac{d\Pi_{bi}}{dM_{bi}}\right|_{(A8)} &= \frac{x_{bi}}{(\sigma-1)} - k_{xb} + \tilde{x}_{bi}(p_{bi}-1) + \left[\tilde{x}_{bi}M_{bi} + M_{bi}(p_{bi}-1)\left.\frac{d\tilde{x}_{bi}}{dp_{bi}}\right|_{(A8)}\right]\left.\frac{dp_{bi}}{dM_{bi}}\right|_{(A8)} \\
&= \frac{x_{bi}}{(\sigma-1)} - k_{xb} + \tilde{x}_{bi}(p_{bi}-1) + \left[1-\sigma\left(\frac{p_{bi}-1}{p_{bi}}\right)\right]\frac{\tilde{x}_{bi}p_{bi}}{(\sigma-1)} \\
&= \frac{x_{bi}}{(\sigma-1)} - k_{xb} + \frac{\tilde{x}_{bi}}{(\sigma-1)},
\end{aligned} \tag{A9}$$

where the second equality follows by computing $\partial\tilde{x}_{bi}/\partial p_{bi}$ from (A5a,b) while keeping $M_{bi}p_{bi}^{1-\sigma}$ constant in the denominator, and by computing dp_{bi}/dM_{bi} from (A8). Setting (A9) equal to zero and using symmetry, we obtain (18). QED

Lemma

Suppose that $N_c = 0$. Then each group will sell inputs to the other groups if and only if,

$$G > \left(\frac{\sigma}{\sigma - 1}\right), \tag{19}$$

in which case the optimal prices are given by:

$$\left(\frac{p_b - 1}{p_b}\right) = \frac{1}{[\sigma + s_{xb}(1-\sigma)]}\left(\frac{G}{G-1}\right). \tag{20}$$

Proof:

Let p_{bi} and q_{bi} denote the prices chosen by group i, with p_{bj} and q_{bj} the prices of the other groups j=1,…,G, j≠i. As p_{bi} is increased, this will raise the costs to the other groups and therefore increase $q_{bj} = \mu_{bj}\,\phi_{bj}$, holding fixed the optimal markups μ_{bj}. Since profits in (1) have been maximized with respect to the markup μ_{bi}, it is legitimate to consider a small change $d\mu_{bi}$ to ensure that the *change in the prices of final goods by all groups are equal.* Specifically, we will choose $d\mu_{bi}$ such that:

$$dq_{bi} = d\mu_{bi}\phi_{bi} = dq_{bj} = \mu_{bj}(\partial\phi_{bj} / \partial p_{bi})dp_{bi}. \tag{A10}$$

The left of this expression if the change in the price of final goods for group i, due to a small change in its markup, and on the right is the change in the final goods price of another group j, due to a change in the price p_{bi} of intermediate inputs sold by group i (but holding the markup of group j fixed at its optimal level). By ensuring that all final goods prices change by the same amount, this ensures that the *relative outputs and market shares of all final* *goods are unchanged.* This will simplify the calculation of the change in group i profits due to the combined change ($d\mu_{bi}$, dp_{bi}) satisfying (A10).

Using symmetry of the initial equilibrium, we divide (A10) by $q_{bi} = q_{bj} = \mu_{bj}\phi_{bj}$, and rewrite this expression as,

$$\frac{dq_{bi}}{dp_{bi}}\frac{1}{q_{bi}} = \frac{dq_{bj}}{dp_{bi}}\frac{1}{q_{bj}} = \frac{\partial\phi_{bj}}{\partial p_{bi}}\frac{1}{\phi_{bj}}. \tag{A10'}$$

Thus, with a rise in p_{bi} leading to equi-proportional increases in q_{bi}, q_{bj}, and ϕ_{bj}, and total consumer expenditure fixed at L, these price increases must be matched with equi-proportional reductions in final goods output y_{bi} and y_{bj}. Thus,

$$\left.\frac{dy_{bi}}{dp_{bi}}\right|_{(A10)}\frac{1}{y_{bi}} = \left.\frac{-dq_{bi}}{dp_{bi}}\right|_{(A10)}\frac{1}{q_{bi}}, \quad i=1,\ldots,G. \tag{A11}$$

Notice that this implies that $q_{bi}\, y_{bi}$ is constant under (A10), as is $\phi_{bj}\, y_{bj}$.

The total change in profits for group i is,

$$\left.\frac{d\Pi_{bi}}{dp_{bi}}\right|_{(A10)} = \frac{\partial\Pi_{bi}}{\partial p_{bi}} + \left.\frac{d\Pi_{bi}}{dq_{bi}}\right|_{(A10)}\left.\frac{dq_{bi}}{dp_{bi}}\right|_{(A10)} + \sum_{j=1,j\neq i}^{G}\left.\frac{d\Pi_{bi}}{dq_{bj}}\right|_{(A10)}\left.\frac{dq_{bj}}{dp_{bi}}\right|_{(A10)}.$$

Using (1), this is evaluated as,

$$\left.\frac{d\Pi_{bi}}{dp_{bi}}\right|_{(A10)} = M_{bi}\left[\tilde{x}_{bi} + (p_{bi}-1)\frac{\partial\tilde{x}_{bi}}{\partial p_{bi}}\right] + N_{bi}\phi_{bi}\left.\frac{dy_{bi}}{dp_{bi}}\right|_{(A10)}. \tag{A12}$$

In the first term of (A12), the elasticity $-(\partial\tilde{x}_{bi}/\partial p_{bi})(p_{bi}/\tilde{x}_{bi})$ is computed from (A5a) (holding $\phi_{bj}\, y_{bj}$ constant) as $[\sigma + s_{xbi}(1-\sigma)]$, where s_{xbi} is the intermediate market share of group i, given by:

$$s_{xbi} = \left[\frac{M_{bi} p_{bi}^{1-\sigma}}{M_{bi} + \sum_{j=1,\, j \neq i}^{G} M_{bj} p_{bj}^{1-\sigma} + \sum_{j=1}^{M_c} p_{cj}^{1-\sigma}} \right].$$

This expression appears as (21) under symmetry.

In the second term of (A12), we note that equi-proportional increases in q_{bi} and reduction in y_{bi} has the effect of holding $q_{bi}\, y_{bi}$ constant in profits, and simply reducing y_{bi} by the amount given by the last term in (A10'). To evaluate this term, differentiate (2) to compute,

$$\frac{\partial \phi_{bj}}{\partial p_{bi}} = p_{bi}^{-\sigma} \left[\frac{\phi_{bj}(1-\beta) M_{bi}}{M_{bj} + \sum_{i=1, i \neq j}^{G} M_{bi} p_{bi}^{1-\sigma} + \sum_{i=1}^{M_c} p_{ci}^{1-\sigma}} \right]$$

$$= \left[\frac{\tilde{x}_b M_b}{y_b (G-1) N_b} \right], \qquad \text{(A13)}$$

where the second line applies with symmetry, with $\tilde{x}_b$ given by (A5a). Then combining (A11)-(A13) and using symmetry, we obtain,

$$\left. \frac{d \Pi_{bi}}{dp_{bi}} \right|_{(A10)} = M_b \tilde{x}_b \left\{ 1 - \left(\frac{p_b - 1}{p_b} \right) [\sigma + s_{xb}(\sigma + 1)] + \frac{1}{(G-1)} \right\}. \qquad \text{(A14)}$$

Setting (A14) = 0, we see that there is a finite solution for p_b if $G > \sigma/(\sigma-1)$, and this solution is (20). Otherwise, expression (A14) remains positive for all positive values of p_b, so the business group optimally chooses $p_b=+\infty$. QED

With these preliminary results, and henceforth using symmetry of the equilibrium, we have the following characterizations of the *V-group* equilibria (Proposition 1) and the *D-group* equilibria (Proposition 2):

Proposition 1

Assume $N_c = 0$. Then the *V-group* equilibria can take one of two forms. Either:

(a) the business groups do not sell inputs to each other ($\tilde{x}_b = 0$), and the number of groups is given by the unique positive solution to,

$$G^2\left(\frac{\Delta\alpha\eta}{L}\right) + G\left[1 - \frac{(\eta-1)\Delta\alpha}{L}\right] - (1+\Delta) = 0\ , \tag{A15}$$

provided that $G \leq \sigma/(\sigma\text{-}1)$, where $\Delta \equiv (\sigma-1)/[(\eta-1)(1-\beta)]$; or,

(b) the business groups do sell to each other ($\tilde{x}_b > 0$), while the number of groups is given by any positive solution to,

$$G^2\left(\frac{\tilde{\Delta}\alpha\eta}{L}\right) + G\left[1 - \frac{(\eta-1)\tilde{\Delta}\alpha}{L}\right] - (1+\tilde{\Delta}) = 0\ , \tag{A16}$$

provided that $G > \sigma/(\sigma\text{-}1)$, where $\tilde{\Delta} \equiv [(\sigma / f(p_b)) - 1]/[(\eta-1)(1-\beta)]$, and,

$$f(p_b) \equiv 1 - (\sigma-1)\left[\frac{(p_b-1)p_b^{-\sigma}(G-1)}{1+(G-1)p_b^{-\sigma}}\right]. \tag{A17}$$

Proof:

(a) Since $N_c = 0$ then $s_{yb} = 1/G$, so that $s_{yb}/(1-s_{yb}) = 1/(G-1)$. Setting $\Pi_b = 0$, and using $\tilde{x}_b = 0$ with (12) and (14) we obtain,

$$N_b k_{yb} = (G-1)M_b k_{xb} + (G-1)\alpha\,. \tag{A18}$$

Using $N_c = 0$, (A2) and (A4) are simplified as,

$$L = G\left[N_b k_{yb}\left(1+\beta(\eta-1)\right) + M_b k_{xb}\sigma + \alpha\right] + M_c k_{xc}\sigma\,, \tag{A2'}$$

$$L = GN_b k_{yb}\left[\eta + \left(\frac{1}{G-1}\right)\right]. \tag{A4'}$$

Condition (18) becomes $x_b = (\sigma-1)k_{xb}$ since $\tilde{x}_b = 0$, so that using (14) and (A6),

$$(\sigma-1)k_{xb} = \frac{(\eta-1)(1-\beta)k_{yb}N_b}{\left(M_b + M_c p_c{}^{1-\sigma}\right)}\,, \tag{A19}$$

where $p_c = \sigma/(\sigma-1)$.

The above are 4 equations in 4 unknowns- G, N_b, M_b and M_c. Setting (A2') = (A4') to eliminate L, and using (A18) repeatedly to convert terms involving N_b to involve M_b instead, we can derive the quadratic equation,

$$G^2 - G\left[1 + \frac{(\sigma-1)}{(1-\beta)(\eta-1)}\left(1+\frac{\alpha}{M_b k_{xb}}\right)^{-1}\right]$$

$$-\left(\frac{M_c k_{xc}}{M_b k_{xb}}\right)\frac{\sigma}{(1-\beta)(\eta-1)}\left(1+\frac{\alpha}{M_b k_{xb}}\right)^{-1} = 0 \tag{A20}$$

Using (A19) we can solve for (M_c/M_b) as,

$$1+\left(\frac{M_c}{M_b}\right)\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma}=\left[\frac{(\eta-1)(1-\beta)}{(\sigma-1)}\right]\left(\frac{N_b k_{yb}}{M_b k_{yb}}\right)$$

$$=\left[\frac{(\eta-1)(1-\beta)}{(\sigma-1)}\right](G-1)\left(1+\frac{\alpha}{M_b k_{xb}}\right),$$

where the second equality follows from (A18). Substituting this into (A20), we obtain a rather long quadratic equation for G, which has the following two solutions:

(i) $M_c=0$ and $G=1+\left[\frac{(\sigma-1)}{(1-\beta)(\eta-1)}\right]\left(1+\frac{\alpha}{M_b k_{xb}}\right)^{-1}$, or ,

(ii) $M_c>0$ and $G=\left(\frac{k_{xc}}{k_{xb}}\right)\left(\frac{\sigma}{\sigma-1}\right)^{\sigma}>1+\left[\frac{(\sigma-1)}{(1-\beta)(\eta-1)}\right]\left(1+\frac{\alpha}{M_b k_{xb}}\right)^{-1}$.

These solutions are viable equilibria provided that $\tilde{x}_b=0$, so that the group finds it optimal to *not* sell its inputs externally, as assumed above. From the above Lemma, we know that $\tilde{x}_b=0$ if and only if $G \le \sigma/(\sigma\text{-}1)$. This immediately rules out the solution in (ii) whenever k_{xc} is close to k_{xb}. The solution in (i) is viable provided that $G \le \sigma/(\sigma\text{-}1)$.

To simplify the expression in (i), we can substitute (A18) into (A4’) to solve for,

$$\frac{\alpha}{M_b k_{xb}}=\alpha\Big/\left\{\frac{L}{G[\eta(G-1)+1)}-\alpha\right\}. \tag{A21}$$

Substituting (A21) into (i), we obtain the quadratic equation in part (a). Part (b) is proved along with Proposition 2, below. QED

Proposition 2

Assume $N_c = 0$. Then in the *D-group* equilibrium unaffiliated upstream firms are profitable ($M_c > 0$), and the business groups also sell to each other ($\tilde{x}_b > 0$), while the number of groups is given by:

$$G = \left(\frac{k_{xc}}{k_{xb}}\right)\left(\frac{\sigma}{\sigma-1}\right)^{\sigma}\left[1+(G-1)p_b^{-\sigma}\right] \tag{A22}$$

which implies,

$$G = (p_b^{\sigma}-1)\Big/\left[\left(\frac{p_b(\sigma-1)}{\sigma}\right)^{\sigma}\left(\frac{k_{xb}}{k_{xc}}\right)-1\right] . \tag{A23}$$

Proof:

Now we suppose that $G > \sigma/(\sigma\text{-}1)$ so that the group sells externally. Setting $\Pi_b = 0$ and using (14), we obtain:

$$\frac{N_b k_y}{(G-1)} + \tilde{x}_b M_b(p_b - 1) = M_b k_{xb} + \alpha. \tag{A24}$$

The full-employment conditions (A2') and (A4') continue to hold.

Condition (18) is $x_b + \tilde{x}_b = (\sigma-1)k_{xb}$, so that using (A6) and (A5a) we obtain,

$$\tilde{x}_b = (G-1)x_b p_b^{-\sigma} \;\Rightarrow\; x_b = \frac{(\sigma-1)k_{xb}}{\left[1+(G-1)p_b^{-\sigma}\right]} .$$

Substituting this into (A24) we obtain,

$$\frac{N_b k_{yb}}{(G-1)} = M_b k_{xb} f(p_b) + \alpha , \tag{A24'}$$

where $f(p_b)$ is defined in (A17).

Then setting (A2') = (A4') to eliminate L, and repeatedly substituting (A24') to replace terms involving N_b with those involving M_b, we obtain the quadratic equation,

$$G^2 - G\left[1 + \frac{\sigma - f(p_b)}{(1-\beta)(\eta-1)}\left(f(p_b) + \frac{\alpha}{M_b k_{xb}}\right)^{-1}\right]$$
$$-\left(\frac{M_c k_{xc}}{M_b k_{xb}}\right)\frac{\sigma}{(1-\beta)(\eta-1)}\left(f(p_b) + \frac{\alpha}{M_b k_{xb}}\right)^{-1} = 0. \qquad \text{(A25)}$$

When $M_c = 0$, then G is solved as:

$$G = 1 + \frac{\sigma - f(p_b)}{(1-\beta)(\eta-1)}\left(f(p_b) + \frac{\alpha}{M_b k_{xb}}\right)^{-1}. \qquad \text{(A25')}$$

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When $M_c \geq 0$, we first rewrite the expression $(\sigma - 1)k_{xb} = x_b + \tilde{x}_b$ using (A6) and (A5a) as,

$$(\sigma - 1)k_{xb} = \frac{k_{yb} N_b (1-\beta)(\eta-1)\left[(G-1)p_b^{-\sigma} + 1\right]}{\left[M_b + M_b(G-1)p_b^{1-\sigma} + M_c p_c^{1-\sigma}\right]},$$

where $p_c = \sigma/(\sigma\text{-}1)$. It follows using (A24') and (A17) that we can solve for M_c as,

$$\frac{M_c p_c^{1-\sigma}}{\left[1 + (G-1)p_b^{-\sigma}\right]} = \frac{M_b}{(\sigma-1)}\left[(G-1)\left(f(p_b) + \frac{\alpha}{M_b k_{xb}}\right)(\eta-1)(1-\beta) - \sigma + f(p_b)\right]. \qquad \text{(A26)}$$

Notice that when $M_c = 0$ in (A26), we again solve for G as in (A25'). When $M_c > 0$, we substitute (A26) into (A25) to obtain a rather long quadratic equation in G. One solution (for $M_c = 0$) is (A25'), and the other solution (for $M_c > 0$) is given by (A23).

In order to solve completely for all endogenous variables, we can make use of (A4') together with (A24') to obtain,

$$\frac{\alpha}{M_b k_{xb}} = \alpha f(p_b) \Big/ \left\{ \frac{L}{G[\eta(G-1)+1)} - \alpha \right\}. \tag{A27}$$

Substituting (A27) into (A25'), we obtain (A16). Then there are five unknowns – p_b, s_{xb}, G, M_b, and M_c – and five equations to solve for them – (20), (21), (A16)-(A17) or (A23), (A26) and (A27). QED

Propositions 1 and 2 provide us with the equations used to compute the V-group and D-group equilibria. Finally, we turn to the case of *U-groups*, in which case $N_c > 0$. We first need to derive the prices charges by these groups for the external sale of intermediate inputs, assume that the business group cannot discriminate in its sales to other groups or to downstream unaffiliated firms. Then the optimal price for external sales of the intermediate input is:

<u>Derivation of Pricing Equation for U-groups</u>:

With $M_c = 0$ and $N_c \geq 0$, the optimal price p_b will satisfy:

$$\left(\frac{p_b - 1}{p_b}\right) = \left\{ \frac{1 + \dfrac{\theta}{(G-1)}\left[1 + \left(\dfrac{s_{yc}}{1 - s_{yb}}\right)(\lambda - 1)\right]}{\sigma + (1-\sigma)\left[\theta s_{xb} + \dfrac{(1-\theta)}{G}\right]} \right\} \tag{A28}$$

where θ is the fraction of external sales $\tilde{x}_b$ that are sold to other groups, and,

$$\lambda = \left[\frac{1 + (G-1)p_b^{1-\sigma}}{G p_b^{1-\sigma}}\right]. \tag{A29}$$

Proof:

Let p_{bi} and q_{bi} denote the prices chosen by group i, with p_{bj} and q_{bj} the prices of the other groups j=1,…,G, j≠i. As p_{bi} is increased, this will raise the costs to the other groups and therefore increase $q_{bj} = \mu_{bj}\,\phi_{bj}$, holding fixed the optimal markups μ_{bj}. Since profits in (1) have been maximized with respect to the markup μ_{bi}, it is legitimate to consider a small change $d\mu_{bi}$ to ensure that the market shares of all *other* business groups are held constant. Imposing symmetry on the prices $q_{bj} = q_b$ of all other groups j=1,…,G, j≠i, as well as on the number of final products $N_{bj} = N_b$ for j=1,…,G, and the prices of downstream unaffiliated firms $q_{cj} = q_c$ for j=1,…,N_c, then the market share of group j≠i is written from (11) as,

$$s_{yb} = \frac{N_b q_b^{1-\eta}}{\left[N_b q_{bi}^{1-\eta} + (G-1)N_b q_b^{1-\eta} + N_c q_c^{1-\eta}\right]}. \tag{11'}$$

Totally differentiating this expression, we find that s_{yb} is constant provided that,

$$\frac{d\ln q_{bi}}{d\ln p_{bi}} = \left[\left(\frac{1}{s_{yb}}\right) - (G-1)\right]\frac{d\ln q_b}{d\ln p_{bi}} - \left(\frac{s_{yc}}{s_{yb}}\right)\frac{d\ln q_c}{d\ln p_{bi}}, \tag{A30}$$

where,

$$s_{yc} = \frac{N_c q_c^{1-\eta}}{\left[N_{bi} q_{bi}^{1-\eta} + (G-1)N_b q_b^{1-\eta} + N_c q_c^{1-\eta}\right]} \tag{A31}$$

is the combined market share of all downstream unaffiliated firms.

Thus, we will evaluate the total change in profits for group i due to a small change dp_{bi}, assuming that the downstream prices of other groups q_b and unaffiliated firms q_c are adjusted holding their markups μ_b and μ_c fixed at their optimal level. Further, we assume that the markup for group i, μ_{bi}, is adjusted so that dq_{bi} satisfies (A30), i.e. the market shares of all *other* business groups j=1,…,G, j≠i, are constant. Because total consumer expenditure equals L from (7'), when the market shares for the other groups s_{yb} are fixed, then so is expenditure on their products, so that $q_b y_b = \mu_b \phi_b y_b$ are both fixed: the increase in q_b due to the rising input price p_{bi} will be matched by an equi-proportional reduction in y_b. The same is not true for group i, however: the increase in q_{bi} satisfying (A30) will be matched by an reduction in y_{bi} that need not be in the same proportion.

Then the total change in profits for group i is,

$$\frac{d\Pi_{bi}}{d\ln p_{bi}} = \frac{\partial \Pi_{bi}}{\partial \ln p_{bi}} + q_{bi} y_{bi} N_{bi} \left.\frac{d\ln q_{bi}}{d\ln p_{bi}}\right|_{(A30)} + y_{bi} N_{bi} (q_{bi} - \phi_{bi}) \left.\frac{d\ln y_{bi}}{d\ln p_{bi}}\right|_{(A30)} \qquad \text{(A32)}$$

$$= p_{bi} \tilde{x}_{bi} M_{bi} \left[1 + \left(\frac{p_{bi} - 1}{p_{bi}} \right) \frac{\partial \ln \tilde{x}_{bi}}{\partial \ln p_{bi}} \right] + q_{bi} y_{bi} N_{bi} \left.\frac{d\ln q_{bi}}{d\ln p_{bi}}\right|_{(A30)} + y_{bi} N_{bi} (q_{bi} - \phi_{bi}) \left.\frac{d\ln y_{bi}}{d\ln p_{bi}}\right|_{(A30)}$$

The first terms on the right are simply the partial effect on profits of changing the input price p_{bi}, which is evaluated using its elasticity of demand. The other two terms are the effect on profits of changing q_{bi} according to (A30), and the induced effect of all price changes in q_{bi} and q_b on y_{bi}.

In order to evaluate the induced effect on y_{bi}, we write this from (7) as:

$$y_{bi} = \frac{q_{bi}^{-\eta} \; L}{[N_b q_{bi}^{1-\eta} + (G-1) N_b q_b^{1-\eta} + N_c q_c^{1-\eta}]} \;, \tag{7'}$$

where we have made use of symmetry: $q_{bj} = q_b$ for all other groups j=1,…,G, j≠i, $N_{bj} = N_b$ for j=1,…,G, and $q_{cj} = q_c$ for j=1,…,N_c. Totally differentiating (7') and using (A30) we find that,

$$\left.\frac{d\ln y_{bi}}{d\ln p_{bi}}\right|_{(A30)} = \left.\frac{-d\ln q_b}{d\ln p_{bi}}\right|_{(A30)} + \left(\frac{\eta s_{yc}}{s_{yb}}\right)\left(\left.\frac{d\ln q_c}{d\ln p_{bi}}\right|_{(A30)} - \left.\frac{d\ln q_b}{d\ln p_{bi}}\right|_{(A30)}\right). \tag{A33}$$

From the optimal pricing rule (12), and using symmetry so $s_{ybi} = s_{yb}$, we have that,

$$\eta(q_{bi} - \phi_{bi}) = q_{bi} + \left(\frac{s_{yb}}{1 - s_{yb}}\right)\phi_{bi}. \tag{A34}$$

Substituting (A30) and (A33) into (A32), and making use of (A34) along with $Gs_{yb} + s_{yc} = 1$, we obtain:

$$\frac{d\Pi_{bi}}{dp_{bi}} = p_{bi}\tilde{x}_{bi}M_{bi}\left[1 + \left(\frac{p_{bi} - 1}{p_{bi}}\right)\frac{\partial \ln \tilde{x}_{bi}}{\partial \ln p_{bi}}\right]$$

$$+ y_{bi}\phi_{bi}N_{bi}\left[\left.\frac{d\ln q_b}{d\ln p_{bi}}\right|_{(A30)} + \left(\frac{s_{yc}}{1 - s_{yb}}\right)\left(\left.\frac{d\ln q_c}{d\ln p_{bi}}\right|_{(A30)} - \left.\frac{d\ln q_b}{d\ln p_{bi}}\right|_{(A30)}\right)\right]. \tag{A35}$$

To simplify this expression further, use (2), (5) and the constant optimal markups, along with (A5) to compute that,

$$\left.\frac{d\ln q_b}{d\ln p_{bi}}\right|_{(A30)} = \frac{\partial \ln \phi_b}{\partial \ln p_{bi}} = \left[\frac{\tilde{x}_{bb} p_b M_b}{(G-1) y_b \phi_b N_b}\right], \tag{A36}$$

$$\left.\frac{d\ln q_c}{d\ln p_{bi}}\right|_{(A30)} = \frac{\partial \ln \phi_c}{\partial \ln p_{bi}} = \left[\frac{\tilde{x}_{bc} p_b M_b}{y_c \phi_c N_c}\right], \qquad \text{(A37)}$$

each of which can be substituted into (A35). Finally, to compute $(\partial \ln \tilde{x}_{bi} / \partial \ln p_{bi})$ we use (A5) to obtain:

$$\frac{\partial \ln \tilde{x}_{bi}}{\partial \ln p_{bi}} = -\{\sigma + (1-\sigma)[\theta s_{xb} + (1-\theta)/G]\}, \qquad \text{(A38)}$$

where $\theta \equiv (\tilde{x}_{bbi} / \tilde{x}_{bi})$ is the share of a business group's external sales of intermediate inputs this is sold to other groups. Notice that the derivative in (A38) is computed while holding $\phi_b y_b$ constant in the numerator of (A5a), as discussed above. We also treat $\phi_c y_c$ constant in the numerator of (A5b) when computing this elasticity.

Substituting (A36) – (A38) into (A35), and setting the latter equal to zero, we obtain:

$$\left(\frac{p_{bi} - 1}{p_{bi}}\right) = \left\{ \frac{1 + \dfrac{\theta}{(G-1)}\left[1 - \left(\dfrac{s_{yc}}{1 - s_{yb}}\right)\right] + (1-\theta)\left(\dfrac{N_b k_{yb}}{N_c k_{yc}}\right)\left(\dfrac{s_{yc}}{1 - s_{yb}}\right)}{\sigma + (1-\sigma)\left[\theta s_{xb} + \dfrac{(1-\theta)}{G}\right]} \right\}. \qquad \text{(A39)}$$

To simplify this further, note that (1-θ)/θ is the *supply* from each business group to all downstream unaffiliated firms, relative to all other business groups. This must equal the *demand* from downstream unaffiliated firms relative to that from business groups, given by $(\tilde{x}_{bc} / \tilde{x}_{bb}) = \lambda(N_c k_{yc} / N_b k_{yb})$, using (A5), (9), (14) and (A29). Thus,

$$\left(\frac{1-\theta}{\theta}\right) = \frac{\lambda N_c k_{yc}}{(G-1) N_b k_{yb}}. \qquad \text{(A40)}$$

Substituting (A40) into (A39), we obtain (A28). QED

In the denominator of (A28), s_{xb} is still given by (21) but with $M_c = 0$, and is interpreted as the share of total demand for intermediates by a group (including internal demand) coming from *one* other group. We could analogously define the share of total demand for intermediates by a unaffiliated downstream firm supplied by *one* group, which is simply (1/G). Thus, the weighted average $[\theta s_{xb} + (1-\theta)/G]$ appearing in the denominator of (A28) can be interpreted as the share of total demand for intermediates supplied by one group, so that the entire denominator is simply the elasticity of demand for the inputs of a group. If the numerator were unity, then (A28) would be a conventional Lerner pricing formula. Instead the numerator exceeds unity, reflecting the fact that when a group sells an input, it will give competing firms a cost advantage, thereby lowering profits in the final goods market. Accordingly, the business group charges a higher price for its inputs than would a firm that is not vertically-integrated across both markets.

With the prices given by (A28), the *U-group* equilibrium is characterized by:

<u>Proposition 3 (U-Group Equilibria)</u>

Assume $M_c = 0$. Then in the *U-group* equilibrium the business groups sell inputs to unaffiliated downstream firms ($N_c > 0$) and to each other ($\tilde{x}_b > 0$), while the number of groups is given by any positive solution to:

$$\frac{\alpha\tilde{\Delta}G^2}{L}\left[\left(\frac{s_{yb}}{1-s_{yb}}\right)+\eta\right]+G\left[1-\left(\frac{s_{yb}}{1-s_{yb}}\right)\tilde{\Delta}+\frac{\alpha\tilde{\Delta}\eta}{L}\left(\frac{N_c k_{yc}}{N_b k_{yb}}\right)\right]+\left(\frac{N_c k_{yc}}{N_b k_{yb}}\right)=0, \qquad \text{(A41)}$$

provided that $G > \sigma/(\sigma\text{-}1)$, where $\tilde{\Delta} \equiv [(\sigma / g(p_b)) - 1]/[(\eta-1)(1-\beta)]$ and,

$$g(p_b) = 1-(\sigma-1)(p_b-1)\left\{\frac{p_b^{-\sigma}[(G-1)+\lambda(N_c k_{yc}/N_b k_{yb})]}{1+p_b^{-\sigma}[(G-1)+\lambda(N_c k_{yc}/N_b k_{yb})]}\right\}. \qquad \text{(A42)}$$

Proof:

Using $\Pi_b = 0$ and (12), we obtain:

$$\left(\frac{s_{yb}}{1-s_{yb}}\right)N_b k_{yb} + \tilde{x}_b M_b(p_b-1) = M_b k_{xb} + \alpha. \qquad \text{(A43)}$$

Using $\tilde{x}_b = \tilde{x}_{bb} + \tilde{x}_{bc}$ from (A5) with $M_c = 0$, together with x_b from (A6) and $x_b + \tilde{x}_b = (\sigma-1)k_{xb}$ from (18), we can derive,

$$M_b \tilde{x}_b(p_b-1) = M_b k_{xb}[1-g(p_b)],$$

where $g(p_b)$ is defined in (A42). Substituting this into (A43) we obtain,

$$\left(\frac{s_{yb}}{1-s_{yb}}\right)N_b k_{yb} = M_b k_{xb} g(p_b) + \alpha. \qquad \text{(A43')}$$

Setting (A2) = (A4) with $M_c = 0$, and substituting (A43') to replace terms involving M_b with those involving N_b, to obtain:

$$G = G\left[\left(\frac{s_{yb}}{1-s_{yb}}\right)-\left(\frac{\alpha}{N_b k_{yb}}\right)\right]\left[\frac{(\sigma/g(p_b))-1}{(\eta-1)(1-\beta)}\right]-\left(\frac{N_c k_{yc}}{N_b k_{yb}}\right). \qquad \text{(A44)}$$

In order to fully determine the equilibrium, we also need to solve for (N_c/N_b). We will make use of the relations,

$$\left(\frac{\phi_c k_{yb}}{\phi_b k_{yc}}\right) = \left(\frac{y_b}{y_c}\right) = \left(\frac{q_b}{q_c}\right)^{-\eta} = \left\{\left(\frac{\phi_b}{\phi_c}\right)\left[1+\frac{1}{\eta}\left(\frac{s_{yb}}{1-s_{yb}}\right)\right]\right\}^{-\eta}. \qquad \text{(A45)}$$

The first equality of (A45) follows from (9) and (14); the second equality from the CES demand system; and the third equality from the pricing formulas (8) and (12). Making use of the first and last expressions we obtain,

$$\left(\frac{\phi_c}{\phi_b}\right) = \left[1 + \frac{1}{\eta}\left(\frac{s_{yb}}{1 - s_{yb}}\right)\right]^{\eta/(\eta-1)} \left(\frac{k_{yb}}{k_{yc}}\right)^{1/(\eta-1)} \tag{A46}$$

and substituting this into the last equality of (A45) we have,

$$\left(\frac{q_c}{q_b}\right) = \left[1 + \frac{1}{\eta}\left(\frac{s_{yb}}{1 - s_{yb}}\right)\right]^{1/(\eta-1)} \left(\frac{k_{yb}}{k_{yc}}\right)^{1/(\eta-1)}. \tag{A47}$$

The shares s_{yb} and s_{yc} of one business group and all unaffiliated firms are related by $Gs_{yb} + s_{yc} = 1$. It follows that (s_{yb}/s_{yc}) equals,

$$\left(\frac{s_{yb}}{1 - Gs_{yb}}\right) = \frac{N_b q_b^{1-\eta}}{N_c q_c^{1-\eta}} = \left(\frac{N_b}{N_c}\right)\left[1 + \frac{1}{\eta}\left(\frac{s_{yb}}{1 - s_{yb}}\right)\right], \tag{A48}$$

where the first equality follows from the CES demand system, and the second from (A47). Rewriting (A48) we obtain,

$$\left(\frac{N_c k_{yc}}{N_b k_{yb}}\right) = \left[\left(\frac{1 - s_{yb}}{s_{yb}}\right) - (G-1)\right]\left[1 + \frac{1}{\eta}\left(\frac{s_{yb}}{1 - s_{yb}}\right)\right]. \tag{A49}$$

To simplify this further, note that λ in (A29) equals $\left(\phi_c/\phi_b\right)^{(\sigma-1)/(1-\beta)}$, using (2) and (5) with $M_c = 0$. It follows from (A46) that,

$$\left(\frac{s_{yb}}{1 - s_{yb}}\right) = \eta\left[\lambda^{\delta}\left(k_{yc}/k_{yb}\right)^{1/\eta} - 1\right], \tag{A50a}$$

where,

$$\delta = \frac{(1-\beta)(\eta-1)}{\eta(\sigma-1)}. \tag{A50b}$$

Substituting (A50) into (A49), we obtain,

$$\left(\frac{N_c k_{yc}}{N_b k_{yc}}\right) = \lambda^{\delta}\left(k_{yc}/k_{yc}\right)^{1/\eta}\left[\frac{1}{\eta}\left(\lambda^{\delta}\left(k_{yc}/k_{yc}\right)^{1/\eta}-1\right)^{-1}-(G-1)\right]. \tag{A51}$$

To solve for the level of N_b and M_b, we can make use of the full-employment conditions (A2) and (A4) together with (9) and (14) to derive,

$$N_b = \left(\frac{L}{k_{yb}}\right)\Big/\left\{G\left[\eta+\left(\frac{s_{yb}}{1-s_{yb}}\right)\right]+\eta\left(\frac{N_c k_{yc}}{N_b k_{yb}}\right)\right\}, \tag{A52}$$

and,

$$M_b = \left(\frac{L}{G k_{xb}\sigma}\right)\left\{1-\left[\frac{(1+\beta(\eta-1))\left(G+\left(N_c k_{yc}/N_b k_{yb}\right)\right)}{G\eta+G(s_{yb}/(1-s_{yb}))+\eta(N_c k_{yc}/N_b k_{yb})}\right]\right\}-\left(\frac{\alpha}{k_{xb}\sigma}\right). \tag{A53}$$

Substituting (A52) into (A44) we obtain the quadratic equation (A41). The equilibrium is now defined by six variables – G, p_b, s_{xb}, θ, λ and (N_c/N_b) – with six equations given by (21), (A28), (A29), (A40), (A41)-(A42) and (A51). QED